Background for Unification

- Terms made from constructors and variables (for the simple first order case)
- Constructors may be applied to arguments (other terms) to make new terms
- Variables and constructors with no arguments are base cases
- Constructors applied to different number of arguments (arity) considered different
- Substitution of terms for variables

Unification Problem

Given a set of pairs of terms ("equations") 
\{(s_1, t_1), (s_2, t_2), \ldots, (s_n, t_n)\}
(the unification problem) does there exist a substitution \( \sigma \) (the unification solution) of terms for variables such that
\( \sigma(s_i) = \sigma(t_i) \), for all \( i = 1, \ldots, n \)?

Unification Algorithm

- **Delete:** if \( s = t \) (they are the same term) then \( \text{Unif}(S) = \text{Unif}(S') \)
- **Decompose:** if \( s = f(q_1, \ldots, q_m) \) and 
  \( t = g(r_1, \ldots, r_n) \) if \( f = g, m = n \), then
  \( \text{Unif}(S) = \text{Unif}({\{(q_1, r_1), \ldots, (q_m, r_n)\}} \cup S') \)
  else **fail**!
- **Orient:** if \( t = x \) is a variable, and \( s \) is not a variable, \( \text{Unif}(S) = \text{Unif}({\{(x = s)\}} \cup S') \)
- **Eliminate:** if \( s = x \) is a variable, then if \( x \) does not occur in \( t \) (the occurs check), then
  - Let \( \varphi = x \rightarrow t \)
  - \( \text{Unif}(S) = \text{Unif}(\varphi(S')) \circ (x \rightarrow t) \)
  - Let \( \psi = \text{Unif}(\varphi(S')) \)
  - \( \text{Unif}(S) = (x \rightarrow \psi(t)) \circ \psi \)
  - Note: \( \{x \rightarrow a\} \circ \{y \rightarrow b\} = \{y \rightarrow ((x \rightarrow a)(b))\} \circ \{x \rightarrow a\} \) if \( y \) not in \( a \)
  else **fail** (because of occurs check failure)
Example

- $x,y,z$ variables, $f,g$ constructors
- Unify $\{(f(x) = f(g(f(z),y))), (g(y,y) = x)\} = ?$

Example

- $x,y,z$ variables, $f,g$ constructors
- $S = \{(f(x) = f(g(f(z),y))), (g(y,y) = x)\}$ is nonempty
- Unify $\{(f(x) = f(g(f(z),y))), (g(y,y) = x)\} = ?$

Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(g(y,y) = x)$
- Orient: $(x = g(y,y))$
- Unify $\{(f(x) = f(g(f(z),y))), (x = g(y,y))\}$ by Orient

Example

- $x,y,z$ variables, $f,g$ constructors
- Unify $\{(f(x) = f(g(f(z),y))), (x = g(y,y))\} = ?$

Example

- $x,y,z$ variables, $f,g$ constructors
- $\{(f(x) = f(g(f(z),y))), (x = g(y,y))\}$ is nonempty
- Unify $\{(f(x) = f(g(f(z),y))), (x = g(y,y))\} = ?$
Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(x = g(y,y))$
- Unify $\{(f(x) = f(g(f(z),y))), (x = g(y,y))\} = ?$

Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(x = g(y,y))$
- Eliminate $x$ with substitution $\{x \rightarrow g(y,y)\}$
  - Check: $x$ not in $g(y,y)$
- Unify $\{(f(x) = f(g(f(z),y))), (x = g(y,y))\} = ?$

Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(f(g(y,y)) = f(g(f(z),y)))$
- Unify $\{(f(g(y,y)) = f(g(f(z),y)))\}$
  - $\{x \rightarrow g(y,y)\} = ?$

Example

- $x,y,z$ variables, $f,g$ constructors
- $\{(f(g(y,y)) = f(g(f(z),y)))\}$ is non-empty
- Unify $\{(f(g(y,y)) = f(g(f(z),y)))\}$
  - $\{x \rightarrow g(y,y)\} = ?$

Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(f(g(y,y)) = f(g(f(z),y)))$
- Unify $\{(f(g(y,y)) = f(g(f(z),y)))\}$
  - $\{x \rightarrow g(y,y)\} = ?$
Example

- \(x, y, z\) variables, \(f, g\) constructors
- Pick a pair: \((f(g(y,y)) = f(g(f(z),y)))\)
- Decompose: \((f(g(y,y)) = f(g(f(z),y)))\)
  becomes \(\{(g(y,y) = g(f(z,y)))\}\)

- Unify \(\{(f(g(y,y)) = f(g(f(z),y)))\}\)
  \(\circ\ \{x \rightarrow g(y,y)\} = \) ?

Example

- \(x, y, z\) variables, \(f, g\) constructors
- \(\{(g(y,y) = g(f(z),y))\}\) is non-empty

- Unify \(\{(g(y,y) = g(f(z),y))\}\) \(\circ\ \{x \rightarrow g(y,y)\} = \) ?

Example

- \(x, y, z\) variables, \(f, g\) constructors
- Pick a pair: \((g(y,y) = g(f(z),y))\)

- Unify \(\{(g(y,y) = g(f(z),y))\}\)
  \(\circ\ \{x \rightarrow g(y,y)\} = \) ?

Example

- \(x, y, z\) variables, \(f, g\) constructors
- \(\{(y = f(z)); (y = y)\}\) \(\circ\ \{x \rightarrow g(y,y)\} = \) non-empty

- Unify \(\{(y = f(z)); (y = y)\}\) \(\circ\ \{x \rightarrow g(y,y)\} = \) ?

Example

- \(x, y, z\) variables, \(f, g\) constructors
- \(\{(y = f(z)); (y = y)\}\) \(\circ\ \{x \rightarrow g(y,y)\} = \) non-empty

- Unify \(\{(y = f(z)); (y = y)\}\) \(\circ\ \{x \rightarrow g(y,y)\} = \) ?
Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(y = f(z))$
  - Unify $\{(y = f(z)); (y = y)\} \circ \{x \mapsto g(y,y)\} = ?$

---

Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(y = f(z))$
- Eliminate $y$ with $\{y \mapsto f(z)\}$
  - Unify $\{(y = f(z)); (y = y)\} \circ \{x \mapsto g(f(z), f(z))\} = ?$

---

Example

- $x,y,z$ variables, $f,g$ constructors
- $\{(f(z) = f(z))\}$ is non-empty
  - Unify $\{(f(z) = f(z))\}$
    - $\circ \{y \mapsto f(z); x \mapsto g(f(z), f(z))\} = ?$

---

Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(f(z) = f(z))$
  - Unify $\{(f(z) = f(z))\}$
    - $\circ \{y \mapsto f(z); x \mapsto g(f(z), f(z))\} = ?$

---

Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(f(z) = f(z))$
- Delete
  - Unify $\{(f(z) = f(z))\}$
    - $\circ \{y \mapsto f(z); x \mapsto g(f(z), f(z))\} = ?$
    - Unify $\{\} \circ \{y \mapsto f(z); x \mapsto g(f(z), f(z))\}$
Example

- \( x, y, z \) variables, \( f, g \) constructors

Unify \( \{ \} \circ \{ y \to f(z); x \to g(f(z), f(z)) \} = ? \)

Example

- \( x, y, z \) variables, \( f, g \) constructors
- \( \{ \} \) is empty
- Unify \( \{ \} = \) identity function
- Unify \( \{ y \to f(z); x \to g(f(z), f(z)) \} = \{ y \to f(z); x \to g(f(z), f(z)) \} \)

Example

- Unify \( \{(f(x) = f(g(f(z), y)), (g(y, y) = x)\} = \{ y \to f(z); x \to g(f(z), f(z)) \}\)

\[
\begin{align*}
f(\ x\ ) &= f(g(f(z),\ y)) \\
\to f(g(f(z), f(z))) &= f(g(f(z), f(z))) \\
g(\ y,\ y) &= x \\
\to g(f(z), f(z)) &= g(f(z), f(z))
\end{align*}
\]

Example of Failure: Decompose

- Unify \( \{(f(x, g(y)) = f(h(y), x))\} \)
- Decompose: \( f(x, g(y)) = f(h(y), x) \)
- = Unify \( \{(x = h(y)), (g(y) = x)\} \)
- Orient: \( (g(y) = x) \)
- = Unify \( \{(x = h(y)), (x = g(y))\} \)
- Eliminate: \( (x = h(y)) \)
- Unify \( \{(h(y) = g(y)) \circ \{ x \to h(y) \} \)\)
- Decompose only rule in this case, but Decompose fails!

Example of Failure: Occurs Check

- Unify \( \{(f(x, g(x)) = f(h(x), x))\} \)
- Decompose: \( f(x, g(x)) = f(h(x), x) \)
- = Unify \( \{(x = h(x)), (g(x) = x)\} \)
- Orient: \( (g(x) = x) \)
- = Unify \( \{(x = h(x)), (x = g(x))\} \)
- Eliminate only rule that applies in this case, but Eliminate fails because the occurs check fails.