Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Your turn: num_neg – tail recursive

```
# let num_neg list =
```


Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [ ] -> curr_neg
      | (x::xs) ->
        if x < 0 then
 (num_neg_aux xs _

  in num_neg_aux  ?  ?
```


Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] ->
      | (x :: xs) ->

  in num_neg_aux  ?  ?
```


Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->

  in num_neg_aux  ?  ?
```


Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
        num_neg_aux xs  ?

  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
        num_neg_aux xs
          (if x < 0 then 1 + curr_neg
           else curr_neg)
  in num_neg_aux  ?  ?
```


Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
        num_neg_aux xs
          (if x < 0 then 1 + curr_neg
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  in num_neg_aux list  ?
```


Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
        num_neg_aux xs
          (if x < 0 then 1 + curr_neg
           else curr_neg)
  in num_neg_aux list 0
```


Tail Recursion - length

- How can we write length with tail recursion?

```
let length list =
    let rec length_aux list acc_length =
            match list       accumulated value
             with [ ] -> acc_length
                | (x::xs) ->
                  length_aux xs (1 + acc_length)
    in length_aux list 0
           initial acc value      combing operation
```


Folding

```
# let rec fold_left f a list = match list
  with [] -> a | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
   <fun>
```

$\boxed{\text{fold\_left f a } [x_1; x_2;\ldots;x_n] = \text{f}(\ldots(\text{f } (\text{f a } x_1)\ x_2)\ldots)x_n}$

```
# let rec fold_right f list b = match list
  with [ ] -> b | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
   <fun>
```

$\boxed{\text{fold\_right f } [x_1; x_2;\ldots;x_n] \text{ b} = \text{f } x_1(\text{f } x_2 (\ldots(\text{f } x_n \text{ b})\ldots))}$

Folding

- Can replace recursion by fold_right in any forward primitive recursive definition
  - Primitive recursive means it only recurses on immediate subcomponents of recursive data structure
- Can replace recursion by fold_left in any tail primitive recursive definition

Continuations

- A programming technique for all forms of “non-local” control flow:
  - non-local jumps
  - exceptions
  - general conversion of non-tail calls to tail calls
- Essentially it’s a higher-order function version of GOTO

Continuations

- Idea: Use functions to represent the control flow of a program
- Method: Each procedure takes a function as an extra argument to which to pass its result; outer procedure “returns” no result
- Function receiving the result called a continuation
- Continuation acts as “accumulator” for work still to be done

Continuation Passing Style

- Writing procedures such that all procedure calls take a continuation to which to give (pass) the result, and return no result, is called continuation passing style (CPS)

Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.
- A formalization of non-local control flow in denotational semantics
- Possible intermediate state in compiling functional code

Why CPS?

- Makes order of evaluation explicitly clear
- Allocates variables (to become registers) for each step of computation
- Essentially converts functional programs into imperative ones
  - Major step for compiling to assembly or byte code
- Tail recursion easily identified
- Strict forward recursion converted to tail recursion
  - At the expense of building large closures in heap

Other Uses for Continuations

- CPS designed to preserve order of evaluation
- Continuations used to express order of evaluation
- Can be used to change order of evaluation
- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads

Example

- Simple reporting continuation:

```
# let report x = (print_int x; print_newline( ) );;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```
# let addk (a, b) k = k (a + b);;
val addk : int * int -> (int -> 'a) -> 'a = <fun>
# addk (22, 20) report;;
2
- : unit = ()
```


Simple Functions Taking Continuations

- Given a primitive operation, can convert it to pass its result forward to a continuation
- Examples:

```
# let subk (x, y) k = k(x - y);;
val subk : int * int -> (int -> 'a) -> 'a = <fun>
# let eqk (x, y) k = k(x = y);;
val eqk : 'a * 'a -> (bool -> 'b) -> 'b = <fun>
# let timesk (x, y) k = k(x * y);;
val timesk : int * int -> (int -> 'a) -> 'a = <fun>
```


Nesting Continuations

```
# let add_triple (x, y, z) = (x + y) + z;;
val add_triple : int * int * int -> int = <fun>
# let add_triple (x,y,z)=let p = x + y in p + z;;
val add_triple : int * int * int -> int = <fun>
# let add_triple_k (x, y, z) k =
  addk (x, y) (fun p -> addk (p, z) k );;
val add_triple_k: int * int * int -> (int -> 'a) ->
  'a = <fun>
```


add_three: a different order

- # let add_triple (x, y, z) = x + (y + z);;
- How do we write add_triple_k to use a different order?
- let add_triple_k (x, y, z) k =

add_three: a different order

- # let add_triple (x, y, z) = x + (y + z);;
- How do we write add_triple_k to use a different order?
- let add_triple_k (x, y, z) k =

```
      addk (y,z) (fun r -> addk(x,r) k)
```


Recursive Functions

- Recall:

```
# let rec factorial n =
   if n = 0 then 1 else n * factorial (n - 1);;
 val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```

Terms

- A function is in Direct Style when it returns its result back to the caller.
- A function is in Continuation Passing Style when it, and every function call in it, passes its result to another function.
- Instead of returning the result to the caller, we pass it forward to another function giving the computation after the call.

Recursive Functions

```
# let rec factorial n =
    let b = (n = 0) in (* First computation *)
    if b then 1 (* Returned value *)
    else let s = n – 1 in (* Second computation *)
         let r = factorial s in  (* Third computation *)
         n * r (* Returned value *) ;;
val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```


Recursive Functions

```
# let rec factorialk n k =
   eqk (n, 0)
   (fun b ->  (* First computation *)
    if b then k 1 (* Passed value *)
    else subk (n, 1)  (* Second computation *)
    (fun s -> factorialk s  (* Third computation *)
      (fun r -> timesk (n, r) k))) (* Passed value *)
val factorialk : int -> (int -> 'a) -> 'a = <fun>
# factorialk 5 report;;
120
- : unit = ()
```


Recursive Functions

- To make recursive call, must build intermediate continuation to
  - take recursive value: r
  - build it to final result: n * r
  - And pass it to final continuation:
  - times (n, r) k = k (n * r)

Example: CPS for length

```
let rec length list = match list with [] -> 0
    | (a :: bs) -> 1 + length bs
```

What is the let-expanded version of this?

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let rec length list = match list with [] -> 0
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#let rec length list = match list with [] -> 0
    | (a :: bs) -> let r1 = length bs in 1 + r1
```

What is the CSP version of this?

```
#let rec lengthk list k = match list with [ ] -> k 0
   | x :: xs -> lengthk xs (fun r -> addk (1,r) k);;
val lengthk : 'a list -> (int -> 'b) -> 'b = <fun>
# lengthk [2;4;6;8] report;;
4
- : unit = ()
```


CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
```


CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> let r1 = sum xs  in x + r1;;
```


CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> let r1 = sum xs  in x + r1;;
val sum : int list -> int = <fun>
# let rec sumk list k = match list with [ ] -> k 0
    | x :: xs -> sumk xs  (fun r1 -> addk x r1 k);;
```


CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> let r1 = sum xs  in x + r1;;
val sum : int list -> int = <fun>
# let rec sumk list k = match list with [ ] -> k 0
    | x :: xs -> sumk xs  (fun r1 -> addk (x, r1) k);;
val sumk : int list -> (int -> 'a) -> 'a = <fun>
# sumk [2;4;6;8] report;;
20
- : unit = ()
```

CPS for Higher Order Functions

- In CPS, every procedure / function takes a continuation to receive its result
- Procedures passed as arguments take continuations
- Procedures returned as results take continuations
- CPS version of higher-order functions must expect input procedures to take continuations

Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
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- What is the CPS version of this?

```
#let rec allk (pk, l) k =
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Example: all

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Example: all

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#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

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#let rec allk (pk, l) k = match l with [] ->  k true
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Example: all

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#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
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val all : ('a -> bool) -> 'a list -> bool = <fun>
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- What is the CPS version of this?

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#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) ->
```

Example: all

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#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
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- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
```


Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
          (fun b -> if b then                   else
     )
```


Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
          (fun b -> if b then allk (pk, xs) k else k
false)
val allk : ('a -> (bool -> 'b) -> 'b) * 'a list ->
(bool -> 'b) -> 'b = <fun>
```