Programming Languages and Compilers (CS 421)

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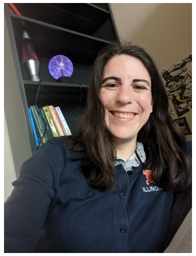

https://courses.grainger.illinois.edu/cs421/fa2023/

Based heavily on slides by Elsa Gunter, which were based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Sign up for makeups!!!

Questions before we start?

Objectives for Today

- We are starting the final part of semantics, which is the last thing we are covering in this class!
- We will cover **Axiomatic Semantics** specifically
- Needed for **WA11**, **final**
- Useful IRL (and shows up on PL/FM quals)

Axiomatic Semantics

- Commonly Floyd-Hoare Logic
  - In practice, often extended
- Based on formal logic (first order predicate calculus)
- Axiomatic Semantics is a logical system built from **axioms** and **inference rules**
- Mainly suited to simple **imperative** programming languages

Questions before we start?

Axiomatic Semantics

Axiomatic Semantics

- Used to formally prove property (**post-condition**) of values of program variables (**state**) after the execution of program, assuming another property (**pre-condition**) of the state holds before execution
- Goal: Derive statements of form

  $$\{P\}\ C\ \{Q\}$$

  - P, Q logical statements about state
  - P precondition, Q postcondition, C program state
- Example: $\{x = 1\}\ x := x + 1\ \{x = 2\}$

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Axiomatic Semantics

- **Approach**: For each type of language statement, give an axiom or inference rule stating how to derive assertions of form

  {P} C {Q}

  where C is a statement of that type
- **Compose** axioms and inference rules to build **proofs** for complex programs

Axiomatic Semantics

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  {P} C {Q}

  where C is a statement of that type
- **Compose** axioms and inference rules to build **proofs** for complex programs

Axiomatic Semantics

- An expression {P} C {Q} is a **partial correctness** statement
- For **total correctness** must also prove that C terminates (i.e. doesn't run forever)
  - Written: [P] C [Q]
- **Will only consider partial correctness here**

Language

We will give rules for **simple imperative language**:

<command> ::=
| <variable> := <term>
| <command>; … ;<command>
| if <statement> then <command> else <command>
| while <statement> do <command> od

(Could add more features, like for-loops.)

Substitution

- **Notation**: P[e / v] (sometimes P[v <- e])
- **Meaning**: Replace every v in P by e
- **Example**: (x + 2) [y - 1 / x] = ((y – 1) + 2)

The Assignment Rule

{P} C {Q}

$$\frac{}{\{P\,[e/x]\}\ x := e\ \{P\}}\ \textbf{ASSIGN}$$

Examples:

$$\frac{}{\{\mathbf{??}\}\ x := y\ \{x = 2\}}\ \textbf{ASSIGN}$$

The Assignment Rule

{P} C {Q}

$$\frac{}{\{P\,[e/x]\,\}\ x := e\ \{P\}}\ \textsf{ASSIGN}$$

Examples:

$$\frac{}{\{\mathbf{??} = \mathbf{2}\}\ x := y\ \{x = \mathbf{2}\}}\ \textsf{ASSIGN}$$

The Assignment Rule

{P} C {Q}

$$\frac{}{\{P\,[e/x]\}\ x := e\ \{P\}}\ \textbf{ASSIGN}$$

Examples:

$$\frac{}{\{\mathbf{y} = 2\}\ \mathbf{x := y}\ \{\mathbf{x} = 2\}}\ \textbf{ASSIGN}$$

The Assignment Rule

{P} C {Q}

$$\frac{}{\{P\,[e/x]\}\; x := e\; \{P\}}\ \text{ASSIGN}$$

Examples:

$$\frac{}{\{\mathbf{y} = 2\}\; \mathbf{x := y}\; \{\mathbf{x} = 2\}}\ \text{ASSIGN}$$

$$\frac{}{\{\mathbf{y} = 2\}\; \mathbf{x := 2}\; \{\mathbf{x} = 2\}}$$

True, but not by this rule

The Assignment Rule

{P} C {Q}

$$\frac{}{\{P\,[e/x]\,\}\ x := e\ \{P\}}\ \textsf{ASSIGN}$$

Examples:

$$\frac{}{\{\mathbf{y} = 2\}\ \mathbf{x := y}\ \{\mathbf{x} = 2\}}\ \textsf{ASSIGN}$$

$$\frac{}{\{\mathbf{2} = 2\}\ \mathbf{x := 2}\ \{\mathbf{x} = 2\}}\ \textsf{ASSIGN}$$

True by this rule

The Assignment Rule

{P} C {Q}

$$\frac{}{\{P[e/x]\}\ x := e\ \{P\}}\ \text{ASSIGN}$$

Examples:

$$\frac{}{\{??\}\ \mathbf{x := x + 1}\ \{\mathbf{x} = n + 1\}}\ \text{ASSIGN}$$

Backwards Reasoning

The Assignment Rule

{P} C {Q}

$$\frac{}{\{P[e/x]\}\ x := e\ \{P\}}\ \text{ASSIGN}$$

Examples:

$$\frac{}{\{??\}\ \mathbf{x := x + 1}\ \{\mathbf{x} = n + 1\}}\ \text{ASSIGN}$$

Weakest Precondition

The Assignment Rule

{P} C {Q}

$$\frac{}{\{P\,[e/x]\,\}\ x := e\ \{P\}}\ \text{ASSIGN}$$

Examples:

$$\frac{}{\{(x = n + 1)[(x + 1)/x]\}\ x := x + 1\ \{x = n + 1\}}\ \text{ASSIGN}$$

Weakest Precondition

The Assignment Rule

{P} C {Q}

$$\frac{}{\{P[e/x]\}\ x := e\ \{P\}}\ \text{ASSIGN}$$

Examples:

$$\frac{}{\{\mathbf{x + 1} = n + 1\}\ \mathbf{x := x + 1}\ \{\mathbf{x} = n + 1\}}\ \text{ASSIGN}$$

Weakest Precondition

The Assignment Rule – Your Turn

$\{P\}\ C\ \{Q\}$

$$\frac{}{\{P\,[e/x]\,\}\ x := e\ \{P\}}\ \text{ASSIGN}$$

What is the weakest precondition of

$x := x + y\ \{x + y = w - x\}$?

The Assignment Rule – Your Turn

{P} C {Q}

$$\frac{}{\{P[e/x]\}\ x := e\ \{P\}}\ \text{ASSIGN}$$

$$\frac{}{\{??\}\ x := x + y\ \{x + y = w - x\}}\ \text{ASSIGN}$$

The Assignment Rule – Your Turn

{P} C {Q}

$$\frac{}{\{\mathbf{P}\,[e/x]\,\}\ x := e\ \{\mathbf{P}\}}\ \text{ASSIGN}$$

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What is P?

The Assignment Rule – Your Turn

{P} C {Q}

$$\frac{}{\{\mathbf{P}\ [e/x]\ \}\ x := e\ \{\mathbf{P}\}}\ \text{ASSIGN}$$

$$\frac{}{\{(\mathbf{x + y = w - x})[\mathbf{??/??}]\}\ x := x + y\ \{\mathbf{x + y = w - x}\}}\ \text{ASSIGN}$$

That is P

The Assignment Rule – Your Turn

{P} C {Q}

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What is e?

The Assignment Rule – Your Turn

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$$\frac{}{\{P[e/\mathbf{x}]\}\ \mathbf{x} := e\ \{P\}}\ \text{ASSIGN}$$

$$\frac{}{\{(x + y = w - x)[(x+y)/\textbf{??}]\}\ x := x + y\ \{x + y = w - x\}}\ \text{ASSIGN}$$

What is x?

The Assignment Rule – Your Turn

{P} C {Q}

$$\frac{}{\{P\,[e/\mathbf{x}]\}\ \mathbf{x} := e\ \{P\}}\ \text{ASSIGN}$$

$$\frac{}{\{(x + y = w - x)[(x + y)/\mathbf{x}]\}\ \mathbf{x} := x + y\ \{x + y = w - x\}}\ \text{ASSIGN}$$

That is x

The Assignment Rule – Your Turn

{P} C {Q}

$$\frac{}{\{P[e/x]\}\ x := e\ \{P\}}\ \text{ASSIGN}$$

$$\frac{}{\{(\mathbf{x} + y = w - \mathbf{x})[(\mathbf{x} + \mathbf{y})/\mathbf{x}]\}\ x := x + y\ \{x + y = w - x\}}\ \text{ASSIGN}$$

Substitute

The Assignment Rule – Your Turn

{P} C {Q}

$$\frac{}{\{P\,[e/x]\,\}\ x := e\ \{P\}}\ \textbf{ASSIGN}$$

$$\frac{}{\{(\mathbf{x+y}) + y = w - (\mathbf{x+y})\}\ x := x + y\ \{x + y = w - x\}}\ \textbf{ASSIGN}$$

Substituted

The Assignment Rule – Your Turn

{P} C {Q}

$$\frac{}{\{P\,[e/x]\,\}\ x := e\ \{P\}}\ \textbf{ASSIGN}$$

$$\frac{}{\{(x+y)+y = w-(x+y)\}\ x := x+y\ \{x+y = w-x\}}\ \textbf{ASSIGN}$$

Done

The Assignment Rule – Your Turn

{P} C {Q}

$$\frac{}{\{P\,[e/x]\}\ x := e\ \{P\}}\ \textbf{ASSIGN}$$

$$\frac{}{\{\mathbf{(x + y) + y = w - (x + y)}\}\ x := x + y\ \{x + y = w - x\}}\ \textbf{ASSIGN}$$

Weakest Precondition

Questions so far?

Strengthening

Precondition Strengthening

{P} C {Q}

$$\frac{P \rightarrow P' \quad \{P'\}\ C\ \{Q\}}{\{P\}\ C\ \{Q\}}\ \textbf{STR}$$

- **Meaning**: If we can show that P implies P′ (P→P′) and we can show that {P′} C {Q}, then we know that {P} C {Q}
- P is **stronger** than P′ means P→P′

Strengthening

Precondition Strengthening

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Examples:

$$\frac{x = 3 \rightarrow x < 7 \qquad \{x < 7\}\ x := x + 3\ \{x < 10\}}{\{x = 3\}\ x := x + 3\ \{x < 10\}}\ \text{STR}$$

Strengthening

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$$\frac{\text{True} \rightarrow 2 = 2 \quad \{2 = 2\}\ x := 2\ \{x = 2\}}{\{\text{True}\}\ x := 2\ \{x = 2\}}\ \text{STR}$$

Strengthening

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$$\frac{x = n \rightarrow x + 1 = n + 1 \quad \{x + 1 = n + 1\}\ x := x + 1\ \{x = n + 1\}}{\{x = n\}\ x := x + 1\ \{x = n + 1\}}\ \text{STR}$$

Strengthening

Questions so far?

Which Inferences are Possible?

??
$$\frac{\{x > 0 \;\&\; x < 5\}\ x := x * x\ \{x < 25\}}{\{x = 3\}\ x := x * x\ \{x < 25\}}\ \text{STR}$$

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✓

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✓

Strengthening

Questions so far?

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Postcondition Weakening

{P} C {Q}

$$\frac{\{P\}\ C\ \{Q'\} \qquad Q' \rightarrow Q}{\{P\}\ C\ \{Q\}}\ \textbf{WEAK}$$

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Example:

$$\frac{\{z = z\ \&\ z = z\}\ x := z;\ y := z\ \{x = z\ \&\ y = z\} \qquad (x = z\ \&\ y = z) \rightarrow (x = y)}{\{z = z\ \&\ z = z\}\ x := z;\ y := z\ \{x = y\}}\ \text{WEAK}$$

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Weakening

Questions so far?

Rule of Consequence

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- Logically equivalent to combination of Precondition **Strengthening** and Postcondition **Weakening**
- Uses $P \rightarrow P'$ and $Q' \rightarrow Q$

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- Logically equivalent to combination of Precondition **Strengthening** and Postcondition **Weakening**
- Uses $P \rightarrow P'$ and $Q' \rightarrow Q$
- Very useful IRL!

Questions so far?

Sequencing

Sequencing

{P} C {Q}

$$\frac{\{P\}\ C_1\ \{Q\} \qquad \{Q\}\ C_2\ \{R\}}{\{P\}\ C_1;\ C_2\ \{R\}}\ \textbf{SEQ}$$

Sequencing

Sequencing

{P} C {Q}

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Sequencing

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Sequencing

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Sequencing

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Questions so far?

Branching

If Then Else

{P} C {Q}

$$\frac{\{P \text{ and } B\}\ C_1\ \{Q\} \quad \{P \text{ and } (\text{not } B)\}\ C_2\ \{Q\}}{\{P\}\ \text{if } B \text{ then } C_1 \text{ else } C_2 \text{ fi}\ \{Q\}}\ \textbf{ITE}$$

Branching

If Then Else

{P} C {Q}

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True branch

Branching

If Then Else

{P} C {Q}

$$\frac{\{P \text{ and } B\}\ C_1\ \{Q\} \quad \{P \text{ and } (\text{not } B)\}\ C_2\ \{Q\}}{\{P\}\ \text{if } B \text{ then } C_1 \text{ else } C_2 \text{ fi}\ \{Q\}}\ \text{ITE}$$

False branch

If Then Else

{P} C {Q}

$$\frac{\{P \text{ and } B\}\ C_1\ \{Q\} \quad \{P \text{ and } (\text{not } B)\}\ C_2\ \{Q\}}{\{P\}\ \text{if } B \text{ then } C_1 \text{ else } C_2 \text{ fi}\ \{Q\}}\ \text{ITE}$$

Example:

$$\frac{\begin{array}{c}\{y = a \ \& \ x < 0\} \\ y := y - x \\ \{y = a + |x|\}\end{array} \quad \begin{array}{c}\{y = a \ \& \ \text{not}\ (x < 0)\} \\ y := y + x \\ \{y = a + |x|\}\end{array}}{\{y = a\}\ \text{if } x < 0 \text{ then } y := y - x \text{ else } y := y + x \text{ fi}\ \{y = a + |x|\}}\ \text{ITE}$$

If Then Else

{P} C {Q}

$$\frac{\{P \text{ and } B\}\ C_1\ \{Q\} \quad \{P \text{ and } (\text{not } B)\}\ C_2\ \{Q\}}{\{P\}\ \text{if } B \text{ then } C_1 \text{ else } C_2 \text{ fi}\ \{Q\}}\ \text{ITE}$$

Example:

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Branching

If Then Else

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$$\frac{\{P \text{ and } B\}\ C_1\ \{Q\} \quad \{P \text{ and } (\text{not } B)\}\ C_2\ \{Q\}}{\{P\}\ \text{if } B \text{ then } C_1 \text{ else } C_2 \text{ fi}\ \{Q\}}\ \text{ITE}$$

Example:

$$\frac{\begin{array}{c}\{y = a \ \& \ x < 0\} \\ y := y - x \\ \{y = a + |x|\}\end{array} \quad \begin{array}{c}\{y = a \ \& \text{ not } (x < 0)\} \\ y := y + x \\ \{y = a + |x|\}\end{array}}{\{y = a\}\ \text{if } x < 0 \text{ then } y := y - x \text{ else } y := y + x \text{ fi}\ \{y = a + |x|\}}\ \text{ITE}$$

Branching

If Then Else

??

{y = a & x < 0}
y := y - x
{y = a + |x|}

??

{y = a & not (x < 0)}
y := y + x
{y = a + |x|}

ITE

{y = a} if x < 0 then y := y - x else y := y + x fi {y = a + |x|}

Branching

If Then Else

$\{y - x = a + |x|\}$

$(y = a \,\&\, x < 0)$ $\quad y := y - x$

$\rightarrow y - x = a + |x|$ $\{y = a + |x|\}$ STR $\qquad$??

$$\frac{\{y = a \,\&\, x < 0\} \;\; y := y - x \;\; \{y = a + |x|\} \qquad \{y = a \,\&\, \text{not}\,(x < 0)\} \;\; y := y + x \;\; \{y = a + |x|\}}{\{y = a\} \text{ if } x < 0 \text{ then } y := y - x \text{ else } y := y + x \text{ fi } \{y = a + |x|\}} \text{ ITE}$$

Branching

If Then Else

??
———————————————
{y - x = a + |x|}

??
———————————————
(y = a & x < 0) → y - x = a + |x|

y := y - x
{y = a + |x|} STR

———————————————
{y = a & x < 0}
y := y - x
{y = a + |x|}

??
———————————————
{y = a & not (x < 0)}
y := y + x
{y = a + |x|}

——————————————— ITE
{y = a} if x < 0 then y := y - x else y := y + x fi {y = a + |x|}

If Then Else

Pure math and logic fragment

$$\frac{x < 0 \rightarrow |x| = -x}{(y = a \,\&\, x < 0) \rightarrow y - x = a + |x|}$$

$$\frac{??}{\{y - x = a + |x|\}}$$

$$y := y - x$$

$$\{y = a + |x|\}_{\textbf{STR}}$$

$$\frac{}{\{y = a \,\&\, x < 0\}}$$

$$y := y - x$$

$$\{y = a + |x|\}$$

$$\frac{??}{\{y = a \,\&\, \text{not } (x < 0)\}}$$

$$y := y + x$$

$$\{y = a + |x|\} \quad \textbf{ITE}$$

$$\frac{}{\{y = a\} \text{ if } x < 0 \text{ then } y := y - x \text{ else } y := y + x \text{ fi } \{y = a + |x|\}}$$

If Then Else

$$x < 0 \rightarrow |x| = -x$$

$$(y = a \,\&\, x < 0) \rightarrow y - x = a + |x|$$

--- ASSIGN

$$\{y - x = a + |x|\}\ y := y - x\ \{y = a + |x|\}$$

--- STR

$$\{y = a \,\&\, x < 0\}\ y := y - x\ \{y = a + |x|\}$$

??

$$\{y = a \,\&\, \text{not } (x < 0)\}\ y := y + x\ \{y = a + |x|\}$$

--- ITE

$$\{y = a\}\ \text{if } x < 0 \text{ then } y := y - x \text{ else } y := y + x \text{ fi}\ \{y = a + |x|\}$$

If Then Else

ASSIGN

not (x < 0) → |x| = x

(y = a & not (x < 0)) → (y + x = a + |x|)

{y + x = a + |x|} y := y + x {y = a + |x|}

x < 0 → |x| = -x

ASSIGN

(y = a & x < 0) → y - x = a + |x|

{y - x = a + |x|} y := y - x {y = a + |x|}

STR

{y = a & x < 0} y := y - x {y = a + |x|}

STR

{y = a & not (x < 0)} y := y + x {y = a + |x|}

ITE

{y = a} if x < 0 then y := y - x else y := y + x fi {y = a + |x|}

Branching

Next Class: Looping

Next Class: Time for Review, Too

ICES: Course Evaluation!

(Please be kind and constructive.
 Please also consider gender biases.)

https://ices.citl.illinois.edu/

Next Class

- **LAST CLASS**
- Please bring questions for review
- Great job!!!
- **MP11 due Tuesday**
- **WA11 due Wednesday**
- All deadlines can be found on **course website**
- Use **office hours** and **class forums** for help