Programming Languages and Compilers (CS 421)

Elsa L Gunter
2112 SC, UIUC

http://courses.engr.illinois.edu/cs421

Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha
Axiomatic Semantics

- Also called Floyd-Hoare Logic
- Based on formal logic (first order predicate calculus)
- Axiomatic Semantics is a logical system built from axioms and inference rules
- Mainly suited to simple imperative programming languages
Axiomatic Semantics

- Used to formally prove a property (post-condition) of the state (the values of the program variables) after the execution of program, assuming another property (pre-condition) of the state holds before execution.
Axiomatic Semantics

- Goal: Derive statements of form
  \[ \{P\} C \{Q\} \]
  - \(P, Q\) logical statements about state,
  - \(P\) precondition, \(Q\) postcondition,
  - \(C\) program

- Example: \(\{x = 1\} x := x + 1 \{x = 2\}\)
Axiomatic Semantics

**Approach**: For each type of language statement, give an axiom or inference rule stating how to derive assertions of form 

\[
\{P\} \ C \ \{Q\}
\]

where \( C \) is a statement of that type

- Compose axioms and inference rules to build proofs for complex programs
Axiomatic Semantics

- An expression \( \{P\} C \{Q\} \) is a *partial correctness* statement.
- For *total correctness* must also prove that \( C \) terminates (i.e. doesn’t run forever).
  - Written: \([P] C [Q]\)
- Will only consider partial correctness here.
Language

- We will give rules for simple imperative language

<command>
  ::= <variable> := <term>
  | <command>; ... ;<command>
  | if <statement> then <command> else <command> fi
  | while <statement> do <command> od

- Could add more features, like for-loops
Substitution

- Notation: \( P[e/v] \) (sometimes \( P[v <- e] \))
- Meaning: Replace every \( v \) in \( P \) by \( e \)
- Example:
  \[
  (x + 2) [y-1/x] = ((y - 1) + 2)
  \]
The Assignment Rule

\[
\{ \text{P[} e/\text{x}] \}\ x := e \ \{ \text{P} \}
\]

Example:

\[
\{ \ ? \ \} x := y \ \{ x = 2 \}
\]
The Assignment Rule

\[ \{ P [e/x] \} \ x := e \ \{ P \} \]

Example:

\[ \{ \_ = 2 \} \ x := y \ \{ \text{\textcolor{red}{x}} = 2 \} \]
The Assignment Rule

\[ \{ P[e/x] \} \ x := e \ { P } \]

Example:

\[ \{ y = 2 \} \ x := y \ { x = 2 } \]
The Assignment Rule

\[ \{P \ [e/x]\ \} \ x := e \ \{P\} \]

Examples:

\[ \{y = 2\} \ x := y \ \{x = 2\} \]

\[ \{y = 2\} \ x := 2 \ \{y = x\} \]

\[ \{x + 1 = n + 1\} \ x := x + 1 \ \{x = n + 1\} \]

\[ \{2 = 2\} \ x := 2 \ \{x = 2\} \]
What is the weakest precondition of
\[ x := x + y \{x + y = w - x}\]?

\[
\{ ? \}
\]

\[ x := x + y \]
\[ \{x + y = w - x\}\]
The Assignment Rule – Your Turn

What is the weakest precondition of
\[ x := x + y \{ x + y = w - x \}? \]

\[ \{(x + y) + y = w - (x + y)\} \]
\[ x := x + y \]
\[ \{ x + y = w - x \} \]
Precondition Strengthening

\[ P \Rightarrow P' \quad \{ P' \} \subseteq \{ Q \} \]
\[ \{ P \} \subseteq \{ Q \} \]

Meaning: If we can show that \( P \) implies \( P' \) (\( P \Rightarrow P' \)) and we can show that \( \{ P' \} \subseteq \{ Q \} \), then we know that \( \{ P \} \subseteq \{ Q \} \)

\( P \) is stronger than \( P' \) means \( P \Rightarrow P' \)
Precondition Strengthening

Examples:

\[ x = 3 \Rightarrow x < 7 \quad \{ x < 7 \} \quad x := x + 3 \quad \{ x < 10 \} \]

\[ \{ x = 3 \} \quad x := x + 3 \quad \{ x < 10 \} \]

\[ \text{True} \Rightarrow 2 = 2 \quad \{ 2 = 2 \} \quad x := 2 \quad \{ x = 2 \} \]

\[ \{ \text{True} \} \quad x := 2 \quad \{ x = 2 \} \]

\[ x = n \Rightarrow x + 1 = n + 1 \quad \{ x + 1 = n + 1 \} \quad x := x + 1 \quad \{ x = n + 1 \} \]

\[ \{ x = n \} \quad x := x + 1 \quad \{ x = n + 1 \} \]
Which Inferences Are Correct?

\[
\{x > 0 \& x < 5 \} \ x := x \times x \ {x < 25}
\]

\[
\{x = 3\} \ x := x \times x \ {x < 25}
\]

\[
\{x > 0 \& x < 5 \} \ x := x \times x \ {x < 25}
\]

\[
\{x * x < 25 \} \ x := x \times x \ {x < 25}
\]

\[
\{x > 0 \& x < 5 \} \ x := x \times x \ {x < 25}
\]
Which Inferences Are Correct?

\[
\begin{align*}
\{x > 0 \land x < 5\} & \quad x := x \times x \quad \{x < 25\} \\
\{x = 3\} & \quad x := x \times x \quad \{x < 25\}
\end{align*}
\]

\[
\begin{align*}
\{x = 3\} & \quad x := x \times x \quad \{x < 25\} \\
\{x > 0 \land x < 5\} & \quad x := x \times x \quad \{x < 25\}
\end{align*}
\]

\[
\begin{align*}
\{x \times x < 25\} & \quad x := x \times x \quad \{x < 25\} \\
\{x > 0 \land x < 5\} & \quad x := x \times x \quad \{x < 25\}
\end{align*}
\]
Sequencing

\[
\begin{align*}
\{P\} & \quad C_1 \quad \{Q\} \quad \{Q\} \quad C_2 \quad \{R\} \\
\{P\} & \quad C_1; \quad C_2 \quad \{R\}
\end{align*}
\]

Example:

\[
\begin{align*}
\{z = z & \land z = z\} & \quad x := z \quad \{x = z & \land z = z\} \\
\{x = z & \land z = z\} & \quad y := z \quad \{x = z & \land y = z\} \\
\{z = z & \land z = z\} & \quad x := z; \quad y := z \quad \{x = z & \land y = z\}
\end{align*}
\]
Sequencing

\[ \{P\} \ C_1 \ \{Q\} \quad \{Q\} \ C_2 \ \{R\} \]
\[ \{P\} \ C_1; \ C_2 \ \{R\} \]

Example:

\[ \{z = z \ & \ z = z\} \ x := z \quad \{x = z \ & \ z = z\} \]
\[ \{x = z \ & \ z = z\} \ y := z \quad \{x = z \ & \ y = z\} \]
\[ \{z = z \ & \ z = z\} \ x := z; \ y := z \quad \{x = z \ & \ y = z\} \]
Postcondition Weakening

\[ \{P\} \ C \ \{Q'\} \quad Q' \rightarrow Q \]

\[ \{P\} \ C \ \{Q\} \]

Example:

\[ \{z = z \land z = z\} \ x := z; \ y := z \ \{x = z \land y = z\} \]

\[ (x = z \land y = z) \rightarrow (x = y) \]

\[ \{z = z \land z = z\} \ x := z; \ y := z \ \{x = y\} \]
Rule of Consequence

\[ P \Rightarrow P' \quad \{P'\} \ C \ {Q'} \quad Q' \Rightarrow Q \]

- Logically equivalent to the combination of Precondition Strengthening and Postcondition Weakening
- Uses \( P \Rightarrow P' \) and \( Q' \Rightarrow Q \)
If Then Else

\[
\begin{align*}
\{P \text{ and } B\} & \; C_1 \{Q\} & \{P \text{ and } (\text{not } B)\} & \; C_2 \{Q\} \\
\{P\} & \text{ if } B \text{ then } C_1 \text{ else } C_2 \text{ fi } \{Q\}
\end{align*}
\]

Example: Want

\[
\{y=a\}
\]

if \(x < 0\) then \(y:= y-x\) else \(y:= y+x\) fi

\[
\{y=a+|x|\}
\]

Suffices to show:

(1) \(\{y=a\&x<0\} \; y:=y-x \; \{y=a+|x|\}\) and

(4) \(\{y=a\&\text{not}(x<0)\} \; y:=y+x \; \{y=a+|x|\}\)
\{y = a \& x < 0\} \; \text{y} := y - x \; \{y = a + |x|\}

(3) \quad (y = a \& x < 0) \Rightarrow y - x = a + |x| \\
(2) \quad \{y - x = a + |x|\} \; \text{y} := y - x \; \{y = a + |x|\} \\
(1) \quad \{y = a \& x < 0\} \; \text{y} := y - x \; \{y = a + |x|\}

(1) Reduces to (2) and (3) by Precondition Strengthening  \\
(2) Follows from assignment axiom  \\
(3) Because \(x < 0 \Rightarrow |x| = -x\)
\{y=a \& \neg(x<0)\} \ y:=y+x \ \{y=a+|x|\}

(6) \ (y=a \& \neg(x<0)) \Rightarrow (y+x=a+|x|) 
(5) \ \{y+x=a+|x|\} \ y:=y+x \ \{y=a+|x|\}

(4) \ \{y=a \& \neg(x<0)\} \ y:=y+x \ \{y=a+|x|\}

(4) Reduces to (5) and (6) by Precondition Strengthening 
(5) Follows from assignment axiom 
(6) Because \(\neg(x<0) \Rightarrow |x| = x\)
If then else

(1) \{y = a \& x < 0\} y := y - x \{y = a + |x|\}

(4) \{y = a \& \text{not}(x < 0)\} y := y + x \{y = a + |x|\}

\{y = a\}

if x < 0 then y := y - x else y := y + x
\{y = a + |x|\}

By the if_then_else rule
While

- We need a rule to be able to make assertions about **while** loops.
  - Inference rule because we can only draw conclusions if we know something about the body
  - Let’s start with:

\[
\begin{align*}
\{ & \mathbf{?} & \} & \text{C} & \{ & \mathbf{?} & \} \\
\{ & \mathbf{?} & \} & \text{while} & B & \text{do} & C & \text{od} & \{ & \mathbf{P} & \}
\end{align*}
\]
The loop may never be executed, so if we want $P$ to hold after, it had better hold before, so let’s try:

$$
\begin{align*}
\{ & \quad ? \quad \} \quad C \quad \{ & \quad ? \quad \} \\
\{ P \} \quad \textbf{while} \quad B \quad \textbf{do} \quad C \quad \textbf{od} \quad \{ P \}
\end{align*}
$$
If all we know is $P$ when we enter the \textbf{while} loop, then we all we know when we enter the body is $(P \text{ and } B)$.

If we need to know $P$ when we finish the \textbf{while} loop, we had better know it when we finish the loop body:

$$
\{ P \text{ and } B \} \; C \; \{ P \}
$$

$$
\{ P \} \; \textbf{while} \; B \; \textbf{do} \; C \; \textbf{od} \; \{ P \}
$$
While

- We can strengthen the previous rule because we also know that when the loop is finished, \textbf{not B} also holds.
- Final \textbf{while} rule:

\[
\begin{align*}
\{ P \text{ and } B \} & \quad C & \quad \{ P \} \\
\{ P \} & \quad \textbf{while} \quad B & \quad \textbf{do} \quad C & \quad \textbf{od} & \quad \{ P \text{ and not } B \}
\end{align*}
\]
While

\[ \{ P \text{ and } B \} \ C \ \{ P \} \]
\[ \{ P \} \text{ while } B \ \textbf{do} \ C \ \textbf{od} \ \{ P \text{ and not } B \} \]

- P satisfying this rule is called a \textit{loop invariant} because it must hold before and after each iteration of the loop.
While rule generally needs to be used together with precondition strengthening and postcondition weakening.

There is NO algorithm for computing the correct $P$; it requires intuition and an understanding of why the program works.
Example

Let us prove

\{x \geq 0 \text{ and } x = a\}

\text{fact} := 1;

\text{while } x > 0 \text{ do } (\text{fact} := \text{fact} \times x; \ x := x - 1) \text{ od}

\{\text{fact} = a!\}
Example

We need to find a condition $P$ that is true both before and after the loop is executed, and such that

$$(P \text{ and not } x > 0) \rightarrow (\text{fact} = a!)$$
Example

- First attempt:
  \[ a! = \text{fact} \times (x!) \]

- Motivation:

- What we want to compute: \( a! \)
- What we have computed: \( \text{fact} \)
  which is the sequential product of \( a \) down through \( (x + 1) \)
- What we still need to compute: \( x! \)
Example

By post-condition weakening suffices to show
1. \{x\geq 0 \text{ and } x = a\}
   
   \text{fact := 1;}
   
   \text{while } x > 0 \text{ do (fact := fact * } x; \ x := x - 1) \text{ od}
   
   \{a! = \text{fact} * (x!) \text{ and not } (x > 0)\}

and

2. \{a! = \text{fact} * (x!) \text{ and not } (x > 0) \} \Rightarrow \{\text{fact} = a!\}
2. \{a! = \text{fact} \times (x!) \text{ and not } (x > 0)\} \Rightarrow \{\text{fact} = a!\}

- Don’t know this if \(x < 0\)
- Need to know that \(x = 0\) when loop terminates
- Need a new loop invariant
- Try adding \(x \geq 0\)
- Then will have \(x = 0\) when loop is done
Example

Second try, combine the two:

\[ P = \{ a! = \text{fact} \times (x!) \text{ and } x \geq 0 \} \]

Again, suffices to show

1. \( \{ x \geq 0 \text{ and } x = a \} \)

   \[
   \text{fact} := 1; \\
   \text{while } x > 0 \text{ do (fact := fact } \times x; x := x - 1) \text{ od} \\
   \{ P \text{ and not } x > 0 \}
   \]

   and

2. \( \{ P \text{ and not } x > 0 \} \Rightarrow \{ \text{fact} = a! \} \)
Example

For 2, we need

\[ \{a! = \text{fact} \times (x!) \text{ and } x \geq 0 \text{ and not } (x > 0)\} \Rightarrow \{\text{fact} = a!\} \]

But \(\{x \geq 0 \text{ and not } (x > 0)\} \Rightarrow \{x = 0\}\) so

\[ \text{fact} \times (x!) = \text{fact} \times (0!) = \text{fact} \]

Therefore

\[ \{a! = \text{fact} \times (x!) \text{ and } x \geq 0 \text{ and not } (x > 0)\} \Rightarrow \{\text{fact} = a!\} \]
For 1, by the sequencing rule it suffices to show

3. \( \{x \geq 0 \text{ and } x = a\}\)
   
   \[
   \text{fact} := 1
   \]
   
   \[
   \{a! = \text{fact} \times (x!) \text{ and } x \geq 0\}\]

And

4. \(\{a! = \text{fact} \times (x!) \text{ and } x \geq 0\}\)
   
   while \(x > 0\) do
   
   \[
   (\text{fact} := \text{fact} \times x; \ x := x - 1)\] od

\[\{a! = \text{fact} \times (x!) \text{ and } x \geq 0 \text{ and not } (x > 0)\}\]
Example

- Suffices to show that

\[
\{ a! = \text{fact} \times (x!) \text{ and } x \geq 0 \}
\]

holds before the while loop is entered and that if

\[
\{ (a! = \text{fact} \times (x!)) \text{ and } x \geq 0 \text{ and } x > 0 \}
\]

holds before we execute the body of the loop, then

\[
\{ (a! = \text{fact} \times (x!)) \text{ and } x \geq 0 \}
\]

holds after we execute the body
Example

By the assignment rule, we have

\{a! = 1 \times (x!) \text{ and } x \geq 0\}

\text{fact} := 1

\{a! = \text{fact} \times (x!) \text{ and } x \geq 0\}

Therefore, to show (3), by precondition strengthening, it suffices to show

\((x \geq 0 \text{ and } x = a) \implies (a! = 1 \times (x!) \text{ and } x \geq 0)\)
Example

\[(x \geq 0 \text{ and } x = a) \Rightarrow (a! = 1 \times (x!) \text{ and } x \geq 0)\]

holds because \(x = a \Rightarrow x! = a!\)

Have that \(\{a! = \text{fact} \times (x!) \text{ and } x \geq 0\}\)
holds at the start of the while loop
Example

To show (4):

\{a! = \text{fact} \times (x!) \text{ and } x \geq 0\}

while \ x > 0 \ do

(fact := fact \times x; \ x := x - 1)

od

\{a! = \text{fact} \times (x!) \text{ and } x \geq 0 \text{ and not } (x > 0)\}

we need to show that

\{(a! = \text{fact} \times (x!)) \text{ and } x \geq 0\}

is a loop invariant
We need to show:
\{(a! = fact * (x!)) \text{ and } x \geq 0 \text{ and } x > 0\}
\begin{center}
\text{( fact = fact * x; x := x – 1 )}
\end{center}
\{(a! = fact * (x!)) \text{ and } x \geq 0\}

We will use assignment rule, sequencing rule and precondition strengthening
By the assignment rule, we have
\[ \{ (a! = \text{fact} \times ((x-1)!)) \text{ and } x - 1 \geq 0 \} \]
\[ x := x - 1 \]
\[ \{ (a! = \text{fact} \times (x!)) \text{ and } x \geq 0 \} \]

By the sequencing rule, it suffices to show
\[ \{ (a! = \text{fact} \times (x!)) \text{ and } x \geq 0 \text{ and } x > 0 \} \]
\[ \text{fact} = \text{fact} \times x \]
\[ \{ (a! = \text{fact} \times ((x-1)!)) \text{ and } x - 1 \geq 0 \} \]
Example

By the assignment rule, we have that
\{(a! = (fact * x) * ((x-1)!)) \text{ and } x - 1 \geq 0\}

\text{fact} = \text{fact} \ast x

\{(a! = \text{fact} \ast ((x-1)!)) \text{ and } x - 1 \geq 0\}

By Precondition strengthening, it suffices to show that
\((a! = \text{fact} \ast (x!)) \text{ and } x \geq 0 \text{ and } x > 0\) \implies
\((a! = (\text{fact} \ast x) \ast ((x-1)!)) \text{ and } x - 1 \geq 0\)
Example

However

\[ \text{fact} \times x \times (x - 1)! = \text{fact} \times (x!) \]

and \((x > 0) \Rightarrow x - 1 \geq 0\)

since \(x\) is an integer, so

\[ \{(a! = \text{fact} \times (x!)) \land x \geq 0 \land x > 0\} \Rightarrow \]

\[ \{(a! = (\text{fact} \times x) \times ((x-1)!)) \land x - 1 \geq 0\} \]
Example

Therefore, by precondition strengthening

\{(a! = \text{fact} \times (x!)) \text{ and } x \geq 0 \text{ and } x > 0\}

\text{fact} = \text{fact} \times x

\{(a! = \text{fact} \times ((x-1)!)) \text{ and } x - 1 \geq 0\}

This finishes the proof