

CS/ECE 374 A ✧ Spring 2026

🌀 Homework 3 🌀

Due Tuesday, February 10, 2026 at 9pm Central Time

1. Prove that the following languages over the alphabet $\Sigma = \{0, 1\}$ are *not* regular.

Recall that a *palindrome* is a string that equals its own reversal.

- (a) $\{0^a 1^b 0^c \mid \text{if } a = b \text{ then } b = 2c\}$
 (b) $\{w w x \mid x, w \in \Sigma^* \text{ and } |w| = |x|\}$
 (c) Even-length binary strings whose first half and second half contain the same number of 1s. More formally:

$$\left\{ w \in \Sigma^* \mid \begin{array}{l} w = xy \text{ for some strings } x \text{ and } y \text{ such that} \\ |x| = |y| \text{ and } \#(1, x) = \#(1, y) \end{array} \right\}$$

- (d) All strings with length $(0 \pmod 4)$ such that half of the string is a palindrome. More formally:

$$\left\{ w \in \Sigma^* \mid \begin{array}{l} w = x z z^R y \text{ for some strings } x, y \text{ and } z \text{ such that} \\ 2|z| = |x| + |y| \end{array} \right\}$$

2. For each of the following languages over the alphabet $\Sigma = \{0, 1\}$, either prove that the language is regular (by constructing and justifying a DFA, NFA, or regular expression) or prove that the language is not regular (for example, using a fooling-set argument).

Recall that a *palindrome* is a string that equals its own reversal. Also recall that Σ^+ denotes the set of all *nonempty* strings over Σ .

- (a) All strings that contain a palindrome of length 374 as a substring. More formally:

$$\left\{ w \in \Sigma^* \mid \begin{array}{l} w = x z y \text{ for some strings } x, y \text{ and } z \text{ such that} \\ z \text{ is a palindrome and } |z| = 374 \end{array} \right\}$$

- (b) All strings in which substrings 00 and 11 occur the same number of times. More formally:

$$\{w \in \Sigma^* \mid \#(00, w) = \#(11, w)\}$$

- (c) All strings in which substrings 01 and 10 occur the same number of times. More formally:

$$\{w \in \Sigma^* \mid \#(01, w) = \#(10, w)\}$$

- (d) All strings x such that (x has 0110 as a substring) and (there exist two distinct indices $i \neq j$ such that i th and $(i + 3)$ rd character of x are the same and j th and $(j + 3)$ rd character of x are the same).

[Note: if x has 0110 as a substring, we already know that there is at least one such index i with the property, but we are asking for at least one more index j with the property. For example, $110110\underline{0110}$ is in the language, and so is $0\underline{1011}0110$.]

*3. Practice only. Do not submit solutions.

A **Moore machine** is a variant of a finite-state automaton that produces output; Moore machines are sometimes called finite-state *transducers*. For purposes of this problem, a Moore machine formally consists of six components:

- A finite set Σ called the input alphabet
- A finite set Γ called the output alphabet
- A finite set Q whose elements are called states
- A start state $s \in Q$
- A transition function $\delta : Q \times \Sigma \rightarrow Q$
- An output function $\omega : Q \rightarrow \Gamma$

More intuitively, a Moore machine is a graph with a special start vertex, where every node (state) has one outgoing edge labeled with each symbol from the input alphabet, and each node (state) is additionally labeled with a symbol from the output alphabet.

The Moore machine reads an input string $w \in \Sigma^*$ one symbol at a time. For each symbol, the machine changes its state according to the transition function δ , and then outputs the symbol $\omega(q)$, where q is the new state. Formally, we recursively define a *transducer* function $\omega^* : Q \times \Sigma^* \rightarrow \Gamma^*$ as follows:

$$\omega^*(q, w) = \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \omega(\delta(q, a)) \cdot \omega^*(\delta(q, a), x) & \text{if } w = ax \end{cases}$$

Given input string $w \in \Sigma^*$, the machine outputs the string $\omega^*(s, w) \in \Gamma^*$. The **output language** $L^\circ(M)$ of a Moore machine M is the set of all strings that the machine can output:

$$L^\circ(M) := \{\omega^*(s, w) \mid w \in \Sigma^*\}$$

- (a) Let M be an arbitrary Moore machine. Prove that $L^\circ(M)$ is a regular language.
- (b) Let M be an arbitrary Moore machine whose input alphabet Σ and output alphabet Γ are identical. Prove that the language

$$L^-(M) = \{w \in \Sigma^* \mid w = \omega^*(s, w)\}$$

is regular. $L^-(M)$ consists of all strings w such that M outputs w when given input w ; these are also called *fixed points* for the transducer function ω^* .

[Hint: These problems are easier than they look!]

Solved problems

4. For each of the following languages, either prove that the language is regular (by constructing an appropriate DFA, NFA, or regular expression) or prove that the language is not regular (using a fooling-set argument).

Recall that a *palindrome* is a string that equals its own reversal: $w = w^R$. Every string of length 0 or 1 is a palindrome.

- (a) Strings in $(0 + 1)^*$ in which no prefix of length at least 2 is a palindrome.

Solution: Regular: $\epsilon + 01^* + 10^*$. Call this language L_a .

Let w be an arbitrary non-empty string in $(0 + 1)^*$. Without loss of generality, assume $w = 0x$ for some string x . There are two cases to consider.

- If x contains a 0, then we can write $w = 01^n0y$ for some integer n and some string y . The prefix 01^n0 is a palindrome of length at least 2. Thus, $w \notin L_a$.
- Otherwise, $x \in 1^*$. Every non-empty prefix of w is equal to 01^n for some non-negative integer $n \leq |x|$. Every palindrome that starts with 0 also ends with 0, so the only palindrome prefixes of w are ϵ and 0, both of which have length less than 2. Thus, $w \in L_a$.

We conclude that $0x \in L_a$ if and only if $x \in 1^*$. A similar argument implies that $1x \in L_a$ if and only if $x \in 0^*$. Finally, trivially, $\epsilon \in L_a$. ■

Rubric: 2½ points = ½ for “regular” + 1 for regular expression + 1 for justification. This is more detail than necessary for full credit.

- (b) Strings in $(0 + 1 + 2)^*$ in which no prefix of length at least 2 is a palindrome.

Solution: Not regular. Call this language L_b .

Consider the set $F = (012)^+$.

Let x and y be arbitrary distinct strings in F .

Then $x = (012)^i$ and $y = (012)^j$ for some positive integers $i \neq j$.

Without loss of generality, assume $i < j$.

Let z be the suffix $(210)^i$.

- $xz = (012)^i(210)^i$ is a palindrome of length $6i \geq 2$, so $xz \notin L_b$.
- $yz = (012)^j(210)^i$ has no palindrome prefixes except ϵ and 0, because $i < j$, so $yz \in L_b$.

Thus, z is a distinguishing suffix for x and y .

We conclude that F is a fooling set for L_b .

Because F is infinite, L_b cannot be regular. ■

Rubric: 2½ points = ½ for “not regular” + 2 for fooling set proof (standard rubric, scaled).

(c) Strings in $(0 + 1)^*$ in which no prefix of length at least 3 is a palindrome.

Solution: Not regular. Call this language L_c .

Consider the set $F = (001101)^+$.

Let x and y be arbitrary distinct strings in F .

Then $x = (001101)^i$ and $y = (001101)^j$ for some positive integers $i \neq j$.

Without loss of generality, assume $i < j$.

Let z be the suffix $(101100)^i$.

- $xz = (001101)^i(101100)^i$ is a palindrome of length $12i \geq 2$, so $xz \notin L_b$.
- $yz = (001101)^j(101100)^i$ has no palindrome prefixes except ϵ and 0 and 00 , because $i < j$, so $yz \in L_b$.

Thus, z is a distinguishing suffix for x and y .

We conclude that F is a fooling set for L_c .

Because F is infinite, L_c cannot be regular. ■

Rubric: 2½ points = ½ for “not regular” + 2 for fooling set proof (standard rubric, scaled).

(d) Strings in $(0 + 1)^*$ in which no *substring* of length at least 3 is a palindrome.

Solution: Regular. Call this language L_d .

Every palindrome of length at least 3 contains a palindrome substring of length 3 or 4. Thus, the complement language $\overline{L_d}$ is described by the regular expression

$$(0 + 1)^*(000 + 010 + 101 + 111 + 0110 + 1001)(0 + 1)^*$$

Thus, $\overline{L_d}$ is regular, so its complement L_d is also regular. ■

Solution: Regular. Call this language L_d .

In fact, L_d is *finite*! Appending either 0 or 1 to any of the underlined strings creates a palindrome suffix of length 3 or 4.

$$\epsilon + 0 + 1 + 00 + 01 + 10 + 11 + 001 + \underline{011} + \underline{100} + 110 + \underline{0011} + \underline{1100}$$

Rubric: 2½ points = ½ for “regular” + 2 for proof:

- 1 for expression for $\overline{L_d}$ + 1 for applying closure
- 1 for regular expression + 1 for justification