More NP Completeness

Lecture 25

April 29, 2025

Part I

Wrap Up 3SAT

Last Time: 3SAT

Recall: last time, we wanted to prove that **3SAT** is **NP**-complete.
Need a function f such that if $\varphi \in$ **CNF-SAT** iff $f(\varphi) \in$ **3SAT**.

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f converts each clause in φ into multiple size-three clauses:

- If $C = (\ell_1)$, include clauses $(\ell_1 \vee x_{C1} \vee x_{C2})$, $(\ell_1 \vee \overline{x_{C1}} \vee x_{C2})$, $(\ell_1 \vee x_{C1} \vee \overline{x_{C2}})$, and $(\ell_1 \vee \overline{x_{C1}} \vee \overline{x_{C2}})$.
- If $C = (\ell_1 \vee \ell_2)$, include clauses $(\ell_1 \vee \ell_2 \vee x_C)$ and $(\ell_1 \vee \ell_2 \vee \overline{x_C})$.
- If $C = (\ell_1 \vee \ell_2 \vee \ell_3)$, include C.
- If $C = (\ell_1 \vee \ldots \vee \ell_k)$ (for $k \geq 4$), include clauses $(\ell_1 \vee \ell_2 \vee x_{C1})$, $(\overline{x_{C1}} \vee \ell_3 \vee x_{C2})$, $\ldots$, $(\overline{x_{C(k-3)}} \vee \ell_{k-1} \vee \ell_k)$.

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Need to show: φ is satisfiable iff $f(\varphi)$ is!

CNF-SAT to 3SAT: If

$\varphi \to f(\varphi)$:

$(\ell_1) \to (\ell_1 \vee x_{C1} \vee x_{C2}) \wedge (\ell_1 \vee \overline{x_{C1}} \vee x_{C2}) \wedge (\ell_1 \vee x_{C1} \vee \overline{x_{C2}}) \wedge (\ell_1 \vee \overline{x_{C1}} \vee \overline{x_{C2}})$

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$(\ell_1 \vee \ell_2 \vee \ell_3) \to (\ell_1 \vee \ell_2 \vee \ell_3)$

$(\ell_1 \vee \ldots \vee \ell_k) \to (\ell_1 \vee \ell_2 \vee x_{C1}) \wedge (\overline{x_{C1}} \vee \ell_3 \vee x_{C2}) \wedge \ldots \wedge (\overline{x_{C(k-3)}} \vee \ell_{k-1} \vee \ell_k)$

$x_{C1} = T$ $\quad x_{C2} = T \ldots x_{C(k-3)} = T$

Claim: If $f(\varphi)$ is satisfiable, so is φ.

Given a satisfying assignment to $f(\varphi)$, keep the values of all the "original" variables.

CNF-SAT to 3SAT: Only-If

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$(\ell_1 \vee \ldots \vee \ell_k) \to (\ell_1 \vee \ell_2 \vee x_{C1}) \wedge (\overline{x_{C1}} \vee \ell_3 \vee x_{C2}) \wedge \ldots \wedge (\overline{x_{C(k-3)}} \vee \ell_{k-1} \vee \ell_k)$

$x_{C1} = T$ $x_{C2} = F$ ... $x_{C(k-3)} = F$

Claim: If φ is satisfiable, so is $f(\varphi)$.

Given a satisfying assignment to φ, construct an assignment to $f(\varphi)$ by keeping all "original" vars

① if # literals is ≤ 3 : assign the added vars arbitrarily

② if # literals is ≥ 4:

Part II

Independent Set

Independent Set

Recall: An independent set is $\boldsymbol{S} \subseteq \boldsymbol{V}$ such that no vertices in $\boldsymbol{S}$ have an edge between them. Let $\boldsymbol{IS} = \{(\boldsymbol{G}, \boldsymbol{k}) \mid \boldsymbol{G}$ has an IS of size $\boldsymbol{k}\}$.

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IS is NP-complete.

IS is in NP: w is the description of an IS of size k.

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Claim

IS *is* ***NP****-complete.*

IS is in ***NP***: w is the description of an IS of size k.

What problem should we reduce to ***IS*** in order to prove hardness?

SAT CNF-SAT 3SAT

3SAT to IS: Intuition

We have a 3SAT formula φ. We want to construct (G, k) such that φ is satisfiable iff G has an IS of size k.

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(We don't need to pick *every* true literal—just don't pick two that contradict!)

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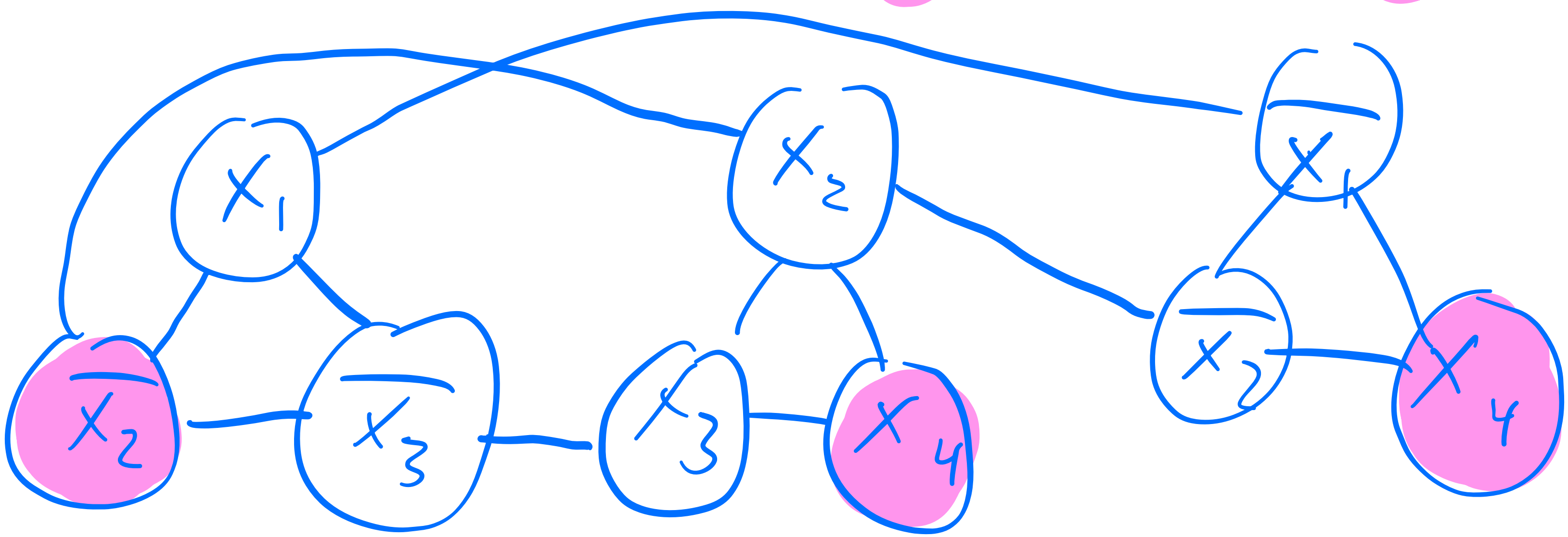
$$\varphi = (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_4)$$



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This reduction clearly runs in polynomial time. (In fact, quadratic.)
Just need to show $\varphi \in \mathbf{3SAT}$ iff $f(\varphi) \in \mathbf{IS}$.

3SAT to IS: Only-If

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Claim: If φ is satisfiable, G has an IS of size k.

Given a satisfying assignment to φ, I know each clause has ≥ 1 true literal

For each clause, pick one literal, and include the corresponding vertex in S.

3SAT to IS: If

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Claim: If G has an IS of size k, φ is satisfiable.

Given an IS $S \subseteq V$ of size k.

① S must pick exactly one vertex from each clause gadget.

② S does not contain vertices labeled x_i & $\overline{x_i}$

$\Rightarrow$ assign variables where $x_i = T$ if x_i is in S, $x_i = F$ if $\overline{x_i}$ is in S, else arbitrary

Related problems

Recall: Independent Set and Clique reduce to each other.

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Claim

$VC = \{(G, k) \mid G$ has a vertex cover of size $k\}$ is NP-complete.

Part III

3-Coloring

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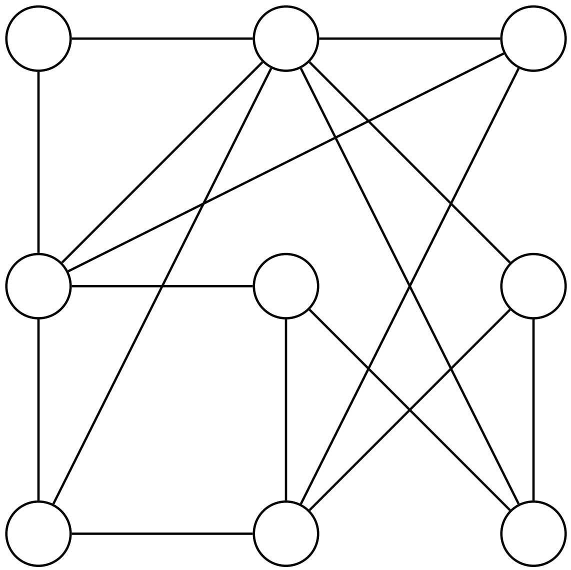

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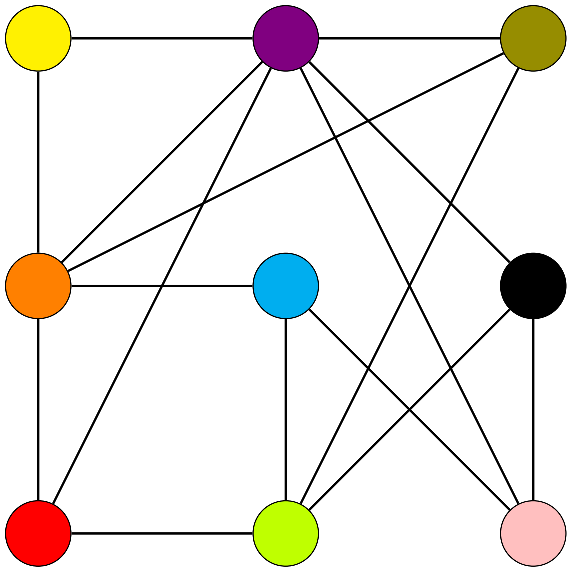

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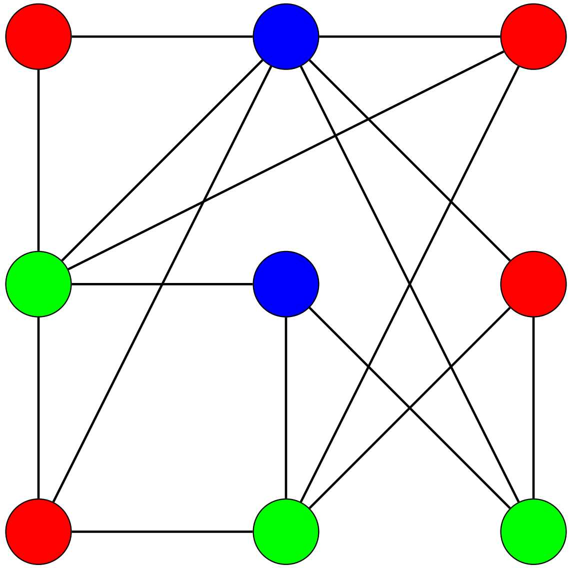

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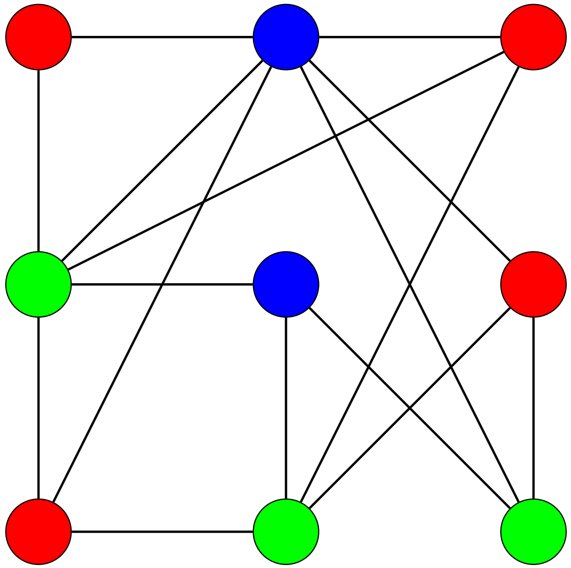

Key question: given a graph G, what is the fewest colors we can use?

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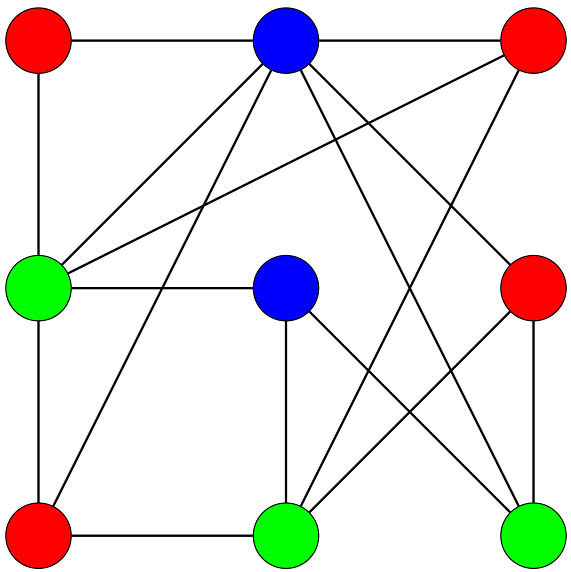

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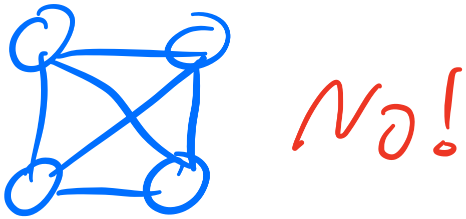
Our focus: Can G be 3-colored?



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3COLOR is in ***NP***: ***w*** is the description of a valid **3**-coloring.

What problem should we reduce to ***3COLOR*** in order to prove hardness?

3SAT

IS (or Clique/VC)

3SAT to 3COLOR: Intuition

We have a **3*SAT*** formula φ. We want to construct a graph G such that φ is satisfiable iff G is 3-colorable.

3SAT to 3COLOR: Intuition

We have a $\mathbf{3}\boldsymbol{SAT}$ formula φ. We want to construct a graph $\boldsymbol{G}$ such that φ is satisfiable iff $\boldsymbol{G}$ is $\mathbf{3}$-colorable.

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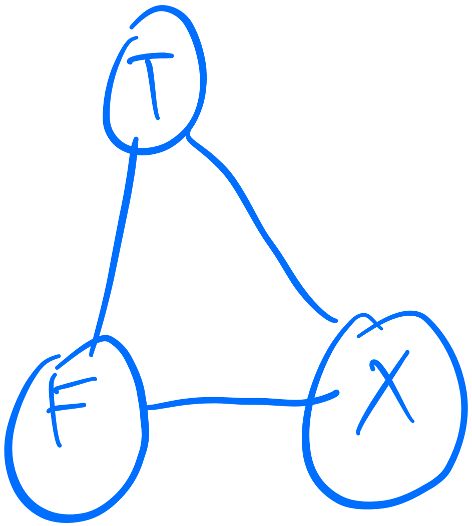

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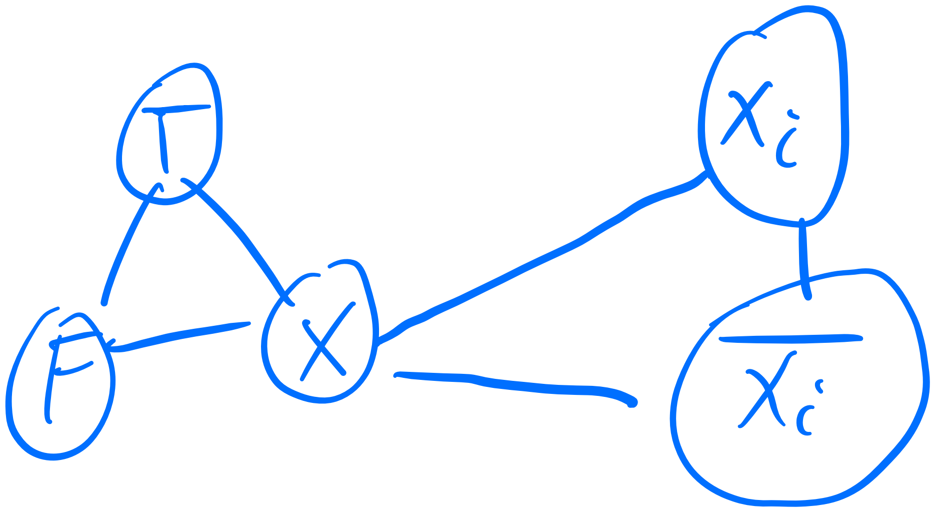

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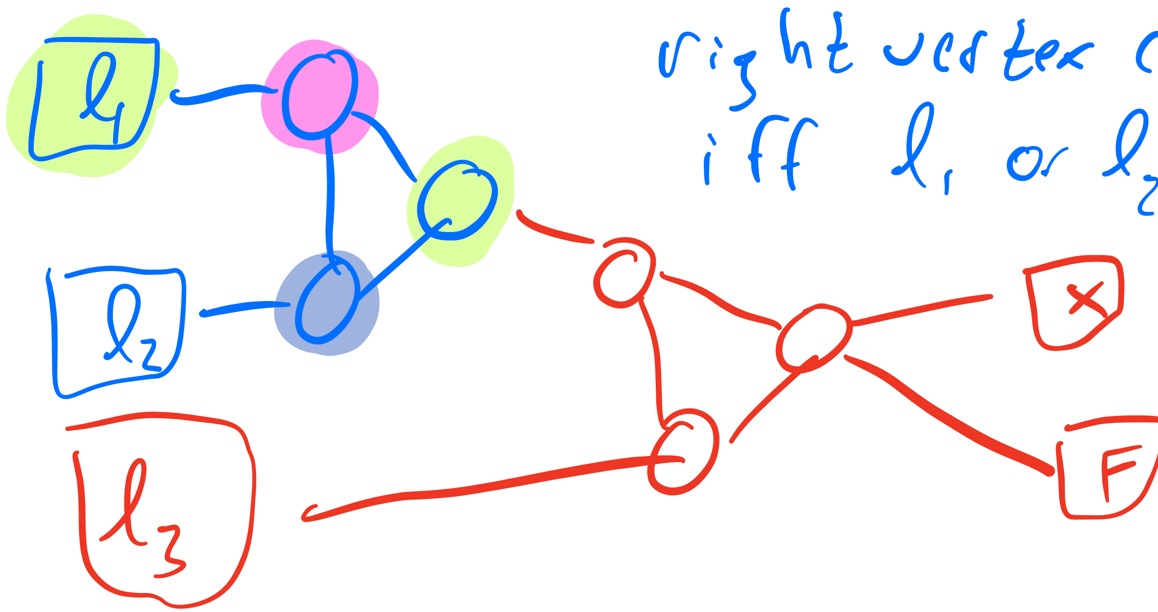
Step 2: ensure each variable is assigned to True or to False.
Step 3: ensure each clause is satisfied.

Simple case: $(l_1 \vee l_2)$ $(l_1 \vee l_2 \vee l_3)$

right vertex can be colored "true" iff l_1 or l_2 is colored "true"



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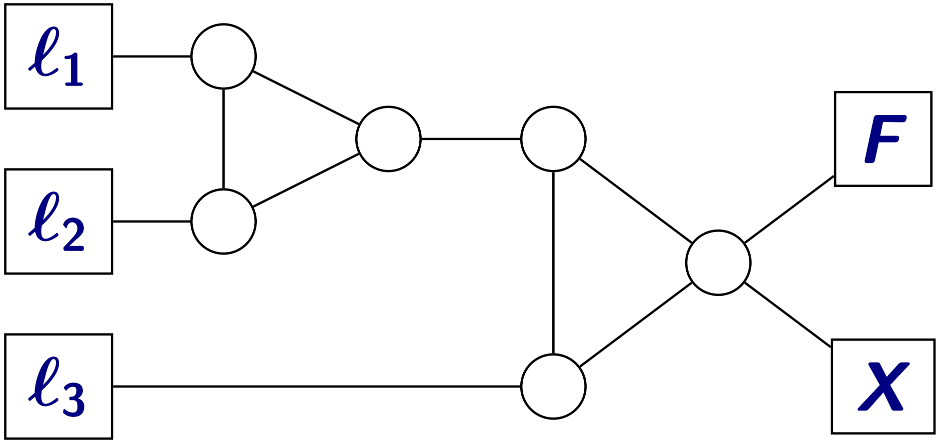

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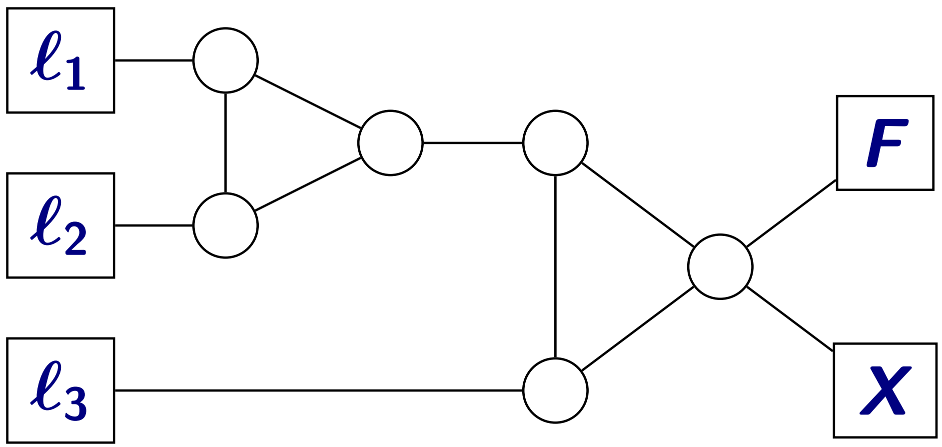
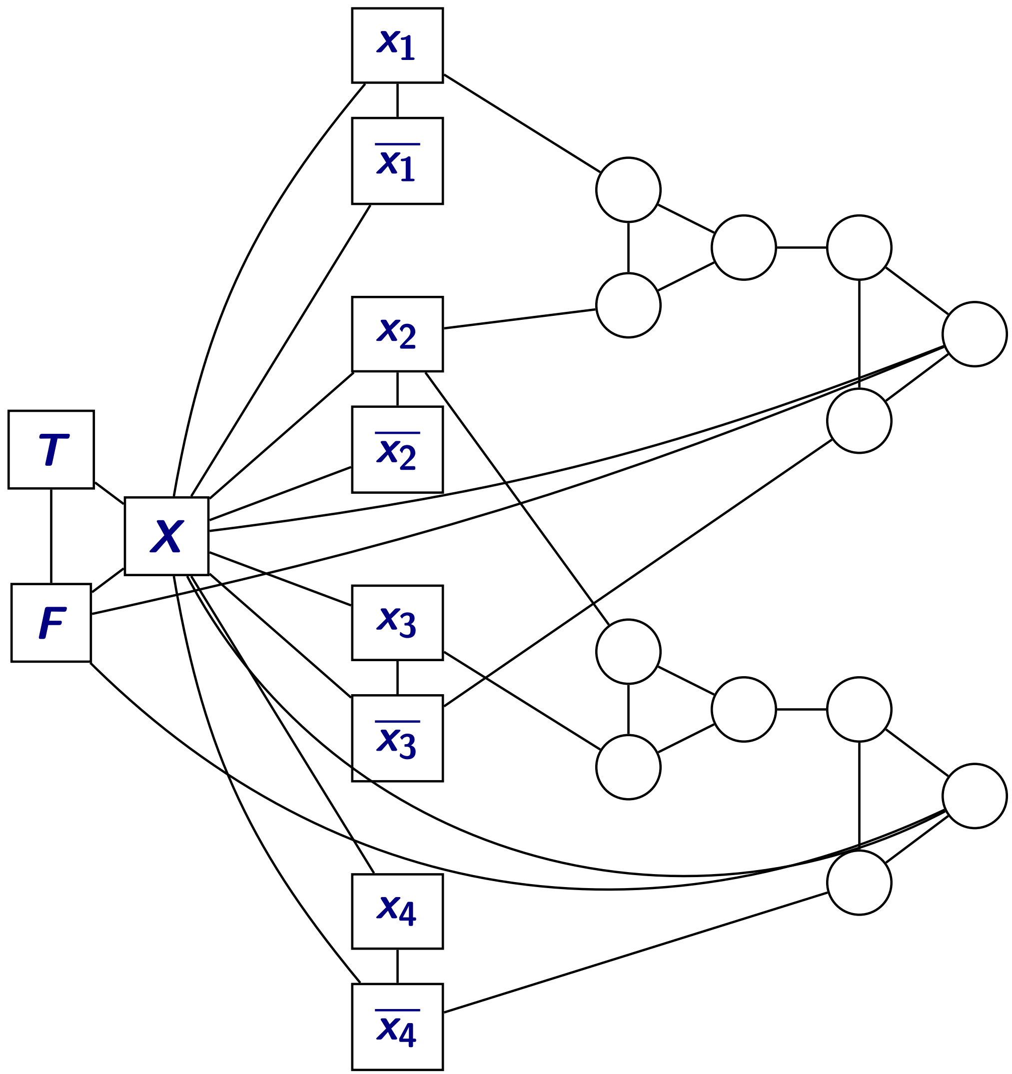


This reduction clearly runs in polynomial time. (In fact, linear.)
Just need to show $\varphi \in 3SAT$ iff $G \in 3COLOR$.

3SAT to 3COLOR: Picture

Say $\varphi = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_2 \vee x_3 \vee \overline{x_4})$



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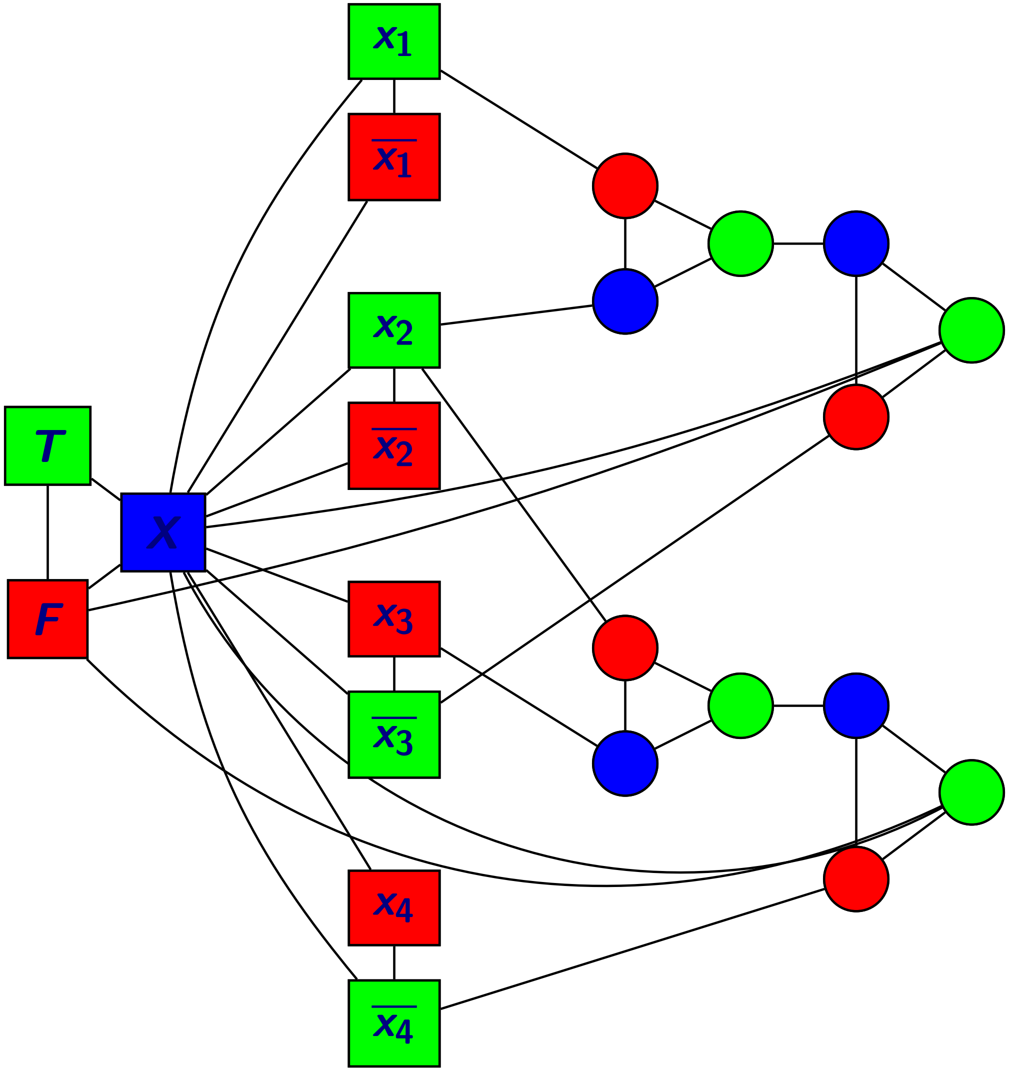

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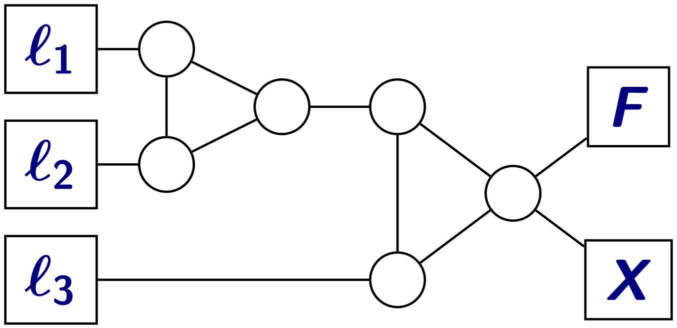

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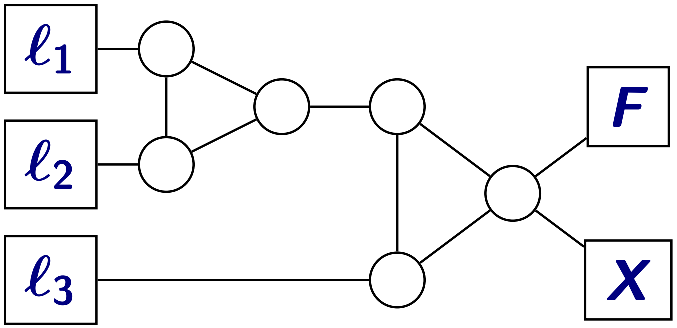

Claim: if G is 3-colorable, φ is satisfiable.

Takeaway Points

Known ***NP***-complete languages

- SAT (from Cook-Levin)
- CNF-SAT (from Cook-Levin)
- 3SAT (from CNF-SAT)
- Independent Set (from 3SAT)
- Clique (from Independent Set)
- Vertex Cover (from Independent Set)
- 3-coloring (from 3SAT)