NP and NP Completeness

Lecture 24

April 24, 2025

Part I

Efficient Computation: P and NP

What is Efficiency?

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Why do we allow for *any* polynomial run time?

- Makes it simpler to describe algorithms.
- Polynomials have helpful closure properties: if $p(n)$ and $q(n)$ are polynomials, so are $p(n) + q(n)$, $p(n) \cdot q(n)$, and $p(q(n))$.
- We are interested in finding problems that *can't* be solved efficiently, so having a lax definition is more meaningful!

NP: Efficient Verification

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We say that a language L is in NP if there is a polynomial $p(\cdot)$ and a machine M (running in time $O(|x|^c)$) such that:

- For every $x \in L$, there is a $w \in \{0,1\}^{p(|x|)}$ such that $M(x, w)$ accepts.
- For every $x \notin L$ and every $w \in \{0,1\}^{p(|x|)}$, $M(x, w)$ rejects.

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Examples we've already seen:

- Independent Set: w describes an IS of size k.
- Clique: w describes a clique of size k.

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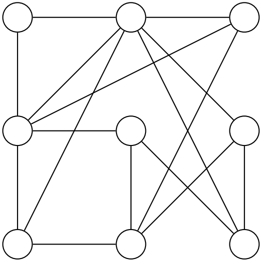

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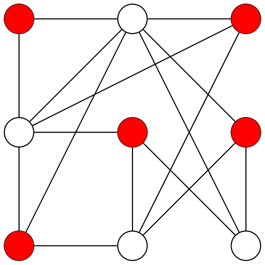

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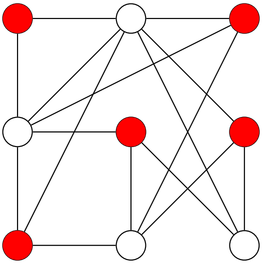

8		6				5		
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7				4				
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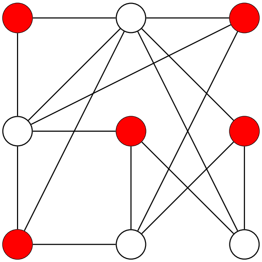

8	4	6	9	3	1	5	7	2
5	9	3	4	7	2	8	1	6
1	2	7	6	8	5	4	9	3
3	5	1	2	9	7	6	8	4
7	6	9	5	4	8	2	3	1
4	8	2	1	6	3	7	5	9
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Can we come up with a *specific* language that isn't in $\boldsymbol{P}$?

Idea: if we can reduce *every* language in $\boldsymbol{NP}$ to some specific language $\boldsymbol{L}$, then we know $\boldsymbol{L}$ in particular is not in $\boldsymbol{P}$!

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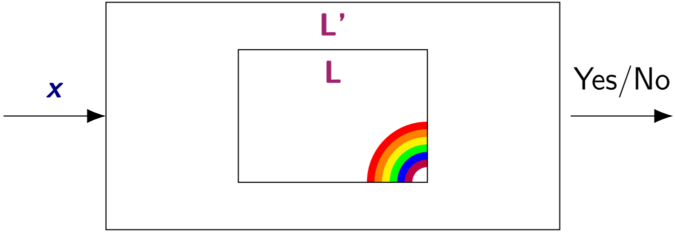

Requirement: as long as "magic box" for L runs in polynomial time, so does the "box" we build for L'.

NP-Hard Example

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DecideL(x):
    Construct a program (machine) P() that:
        For each w ∈ {0,1}^{p(|x|)}, runs M(x, w)
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Better question: is there an NP-hard problem *that is also in* NP? (We call such problems NP-complete.)

Part II

SAT

Boolean Satisfiablility (SAT)

Consider a boolean formula using AND, OR, and NOT:

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Definition

Let $SAT = \{\varphi \mid \varphi \text{ is satisfiable}\}$

Note $SAT \in NP$: we can take w to be an assignment of True/False to each variable!

Using SAT

SAT turns out to be very powerful for modeling other problems!

Example: does $\boldsymbol{G = (V, E)}$ have a *path* that visits every vertex?

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Theorem (Cook-Levin, stronger version)

$\textit{CNF-SAT} = \{\varphi \mid \varphi \text{ is satisfiable and in CNF}\}$ *is* ***NP****-complete.*

This means that (assuming $P \neq NP$) there is no polynomial time algorithm for ***SAT*** nor ***CNF-SAT***!

Part III

3SAT

Using Reductions

How do we prove that more problems are ***NP***-complete?

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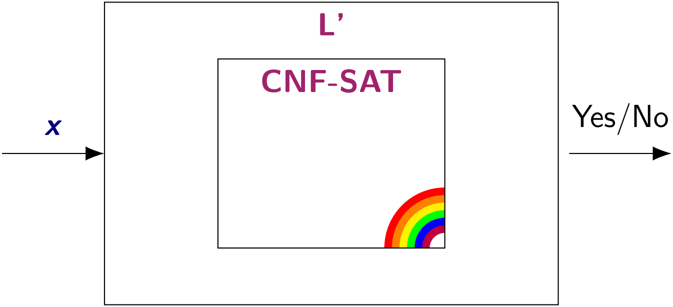

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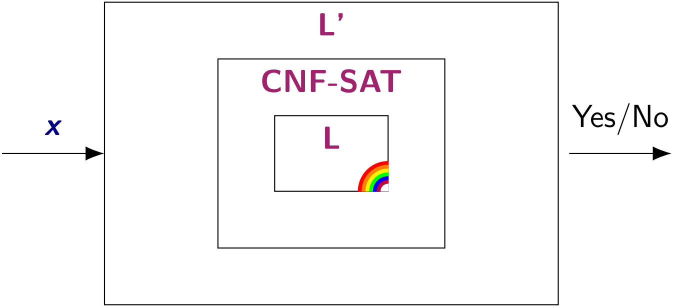

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Common form of reduction: give function $\boldsymbol{f}$ (that can be computed in polynomial time) so that $\boldsymbol{x} \in \boldsymbol{CNF\text{-}SAT}$ iff $\boldsymbol{f(x)} \in \boldsymbol{L}$.

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Given a CNF formula, want to make each clause have 3 literals.

- How do we fix clauses with too few? (eg $(a \vee \overline{b})$)
- How do we fix clauses with too many? (eg $(a \vee \overline{b} \vee c \vee \overline{d})$)

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- If C has three literals, include C in $f(\varphi)$
- If $C = (\ell_1 \vee \ldots \vee \ell_k)$ has $k \geq 4$ literals, include clauses $(\ell_1 \vee \ell_2 \vee x_{C1})$, $(\overline{x_{C1}} \vee \ell_3 \vee x_{C2})$, $\ldots$, $(\overline{x_{C(k-3)}} \vee \ell_{k-1} \vee \ell_k)$ in $f(\varphi)$, where $(x_{C1}, \ldots, x_{C(k-3)})$ are new variables.

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Exercise: formally prove that φ is satisfiable if and only if $f(\varphi)$ is.

- If direction: given a satisfiable assignment for $f(\varphi)$, find a satisfiable assignment for φ
- Only if direction: given a satisfiable assignment for φ, find a satisfiable assignment for $f(\varphi)$

Takeaway Points

Definitions of $\boldsymbol{P}$ and $\boldsymbol{NP}$.

- If $L \in P$, we can efficiently decide if $x \in L$.
- If $L \in NP$, we can efficiently verify proofs that $x \in L$.
- We will assume that $P \neq NP$.

$\boldsymbol{NP}$-hardness and $\boldsymbol{NP}$-completeness

- A problem is NP-hard if *every* problem in NP reduces to it. If it is also in NP itself, we call it NP-complete.
- If you can reduce an NP-hard problem to L in polynomial time, L is also NP-hard.

Known $\boldsymbol{NP}$-complete languages

- SAT
- $CNF\text{-}SAT$
- $3SAT$