Reductions

Lecture 22

April 17, 2025

Course Outline

- Part I: models of computation (reg exps, DFA/NFA, CFGs, TMs)
- Part II: (efficient) algorithm design
- Part III: limits of (efficient) computation
  - Undecidablity: problems that have *no* algorithms
  - NP-Completeness: problems that (we think) have no *efficient* algorithms

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Key tool for proving intractability: reductions!

Part I

Reductions for Algorithms

Recall: Longest Sequences

Longest Increasing Subsequence: Find the longest subsequence of $A[1..n]$ such that each term is larger than the last.

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Longest *Decreasing* Subsequence: Find the longest subsequence of $A[1..n]$ such that each term is *smaller* than the last.

LDS(A) is the same as

LIS(-A)

LDS Reduction

```
from magic import LIS
LDS(A[1..n]):
    Negate every element of A
    Compute seq = LIS(A)
    Negate every element of seq
    Return seq
```

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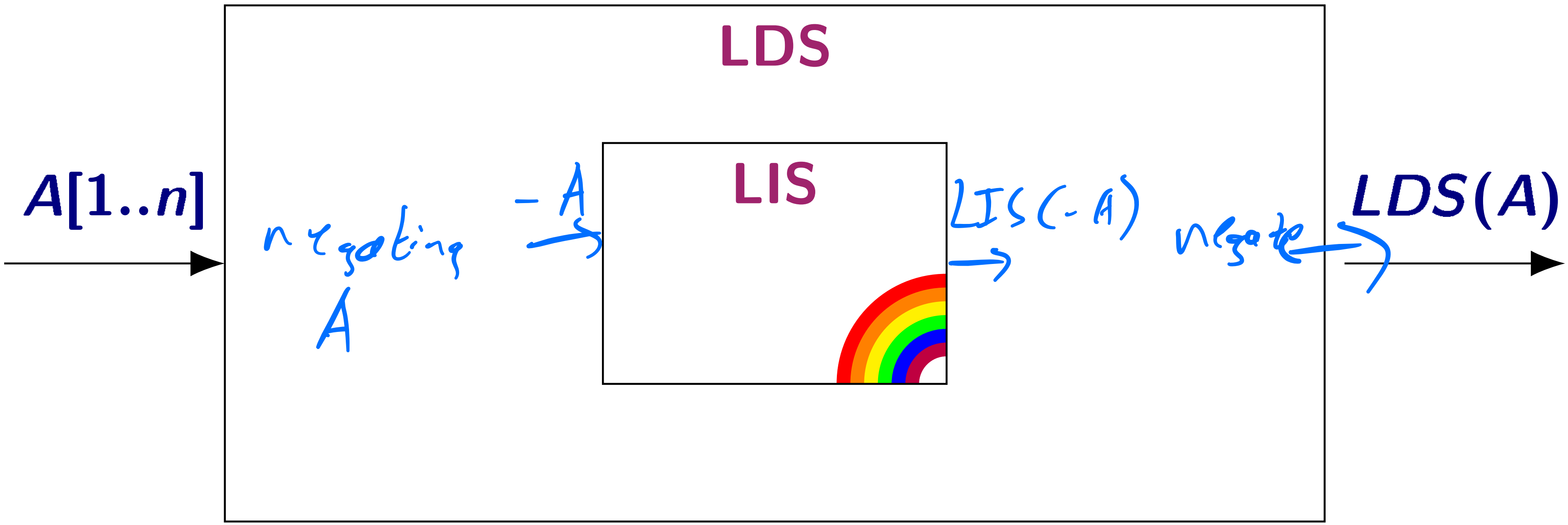
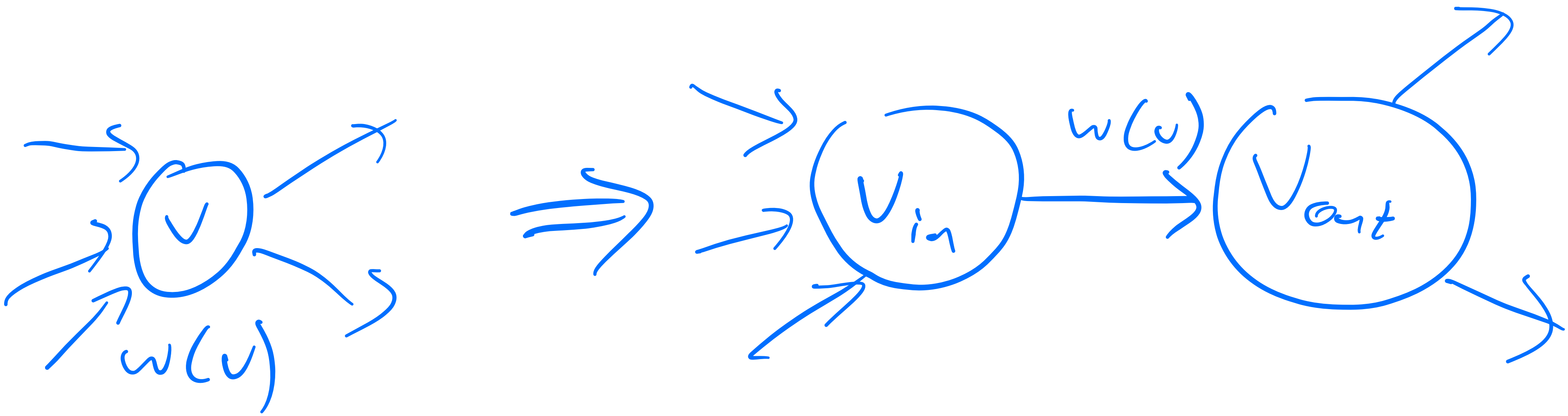


Vertex Weights

Say we have a graph $G = (V, E)$ with weights on the *vertices*. How do we find the length of the shortest path from s to t?



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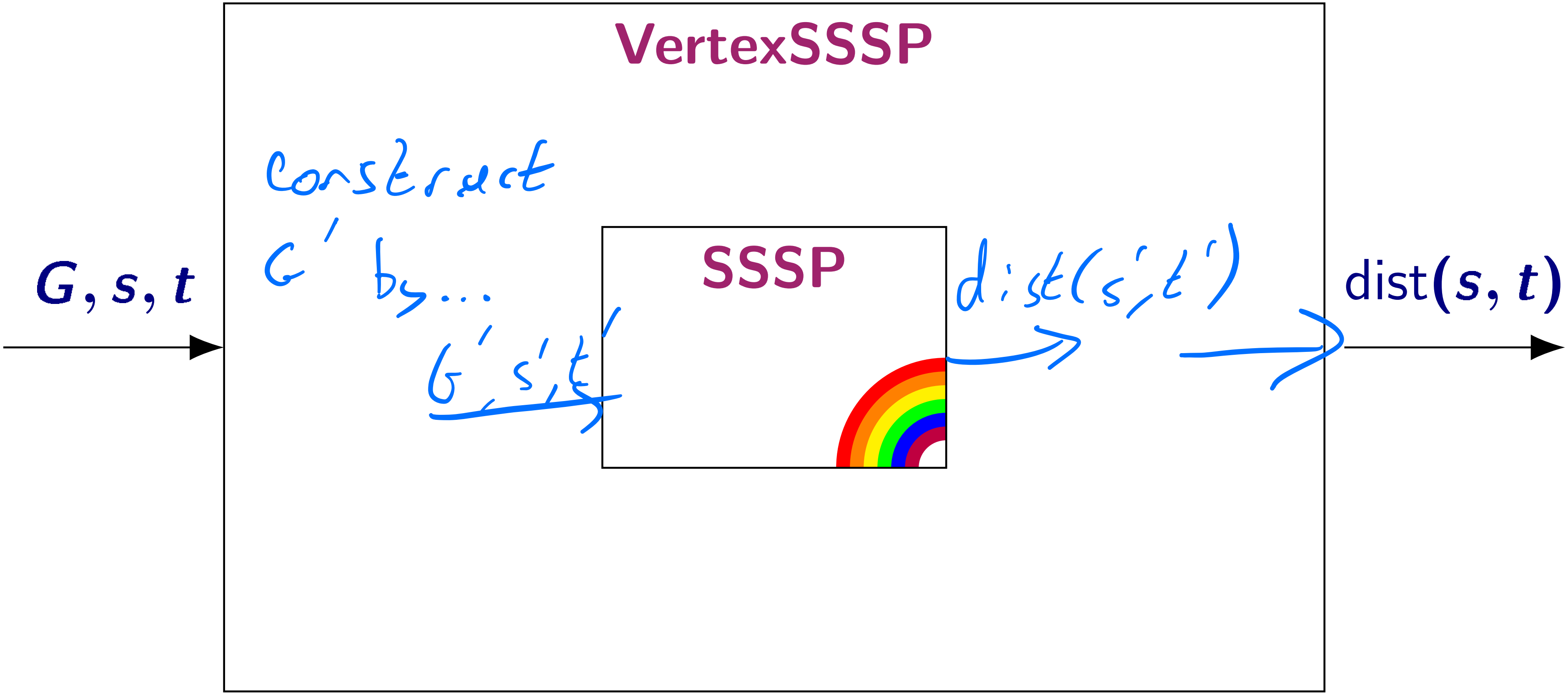

Why Reductions?

Reductions allow us to simplify algorithm design by making use of *abstraction barriers*.

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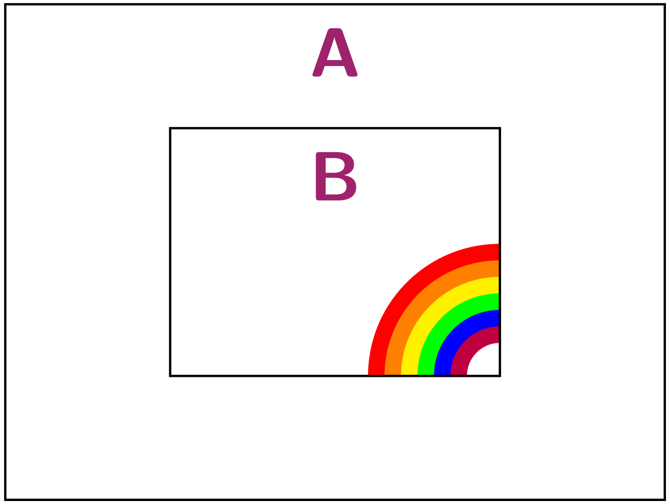

A is "no harder than" ***B***: any algorithm for ***B*** gives one for ***A***.

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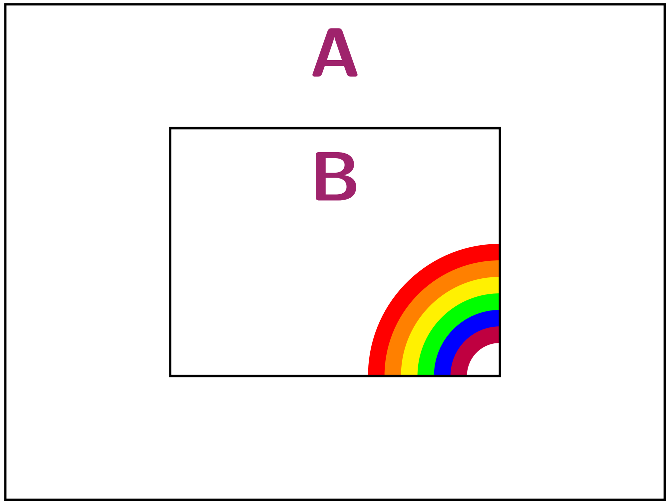

A is "no harder than" B: any algorithm for B gives one for A.

B is "no easier than" A: if A has no "good" algorithm, neither does B!

Part II

Practice with Reductions

Autograding CS 124

Say we've asked a CS 124 student to write a "Hello World!" program.

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We're expecting:

```
main():
    print(‘‘Hello World!’’)
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```
main():
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Can we (algorithmically) check if a student's code works?

Autograding CS 124

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We're expecting:

```
main():
    print(''Hello World!'')
```

Can we (algorithmically) check if a student's code works?

Intuitively simpler question: can we check if a student's code at least doesn't run forever?

Reducing "Hello World!" to Halting

We want to reduce checking if a student's code prints "Hello World!" and halts to just checking if it (eventually) halts.

```
from magic import TestHalt
TestHW(StudentCode):
```

Construct program P st P halts iff SC is correct:

1. run StudentCode
2. if SC didn't print "HelloWorld", loop forever

return TestHalt(P)

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TestHW(StudentCode):
    Create program P that:
        (1) Runs StudentCode
        (2) If the terminal doesn't say ``Hello World!'',
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Note: P halts if and only if `StudentCode` is correct!

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This tells us that checking if a student's code is correct is "no harder than" just checking if it runs forever.

Reducing Halting to "Hello World"

We can use these same ideas to reduce in the opposite direction!

```
from magic import TestHW
TestHalt(StudentCode):
    Create a program P that:
        (1) Runs StudentCode (supressing print statements)
        (2) prints ``Hello World!''
    return TestHW(P)
```

Note: ***P*** will halt and print "Hello World!" if and only if `StudentCode` halts.

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Note: P will halt and print “Hello World!” if and only if `StudentCode` halts.

This means that checking if a student’s code halts is “no harder than” checking if it’s correct—so the two tasks are the same “level of difficulty”!

Independent Set and Clique

Given a graph $G = (V, E)$, we define

- A **clique** as $C \subseteq V$ such that each vertex in C has an edge to *every* other vertex in C.
- A **independent set** as $S \subseteq V$ such that no two vertices in S have an edge between them.

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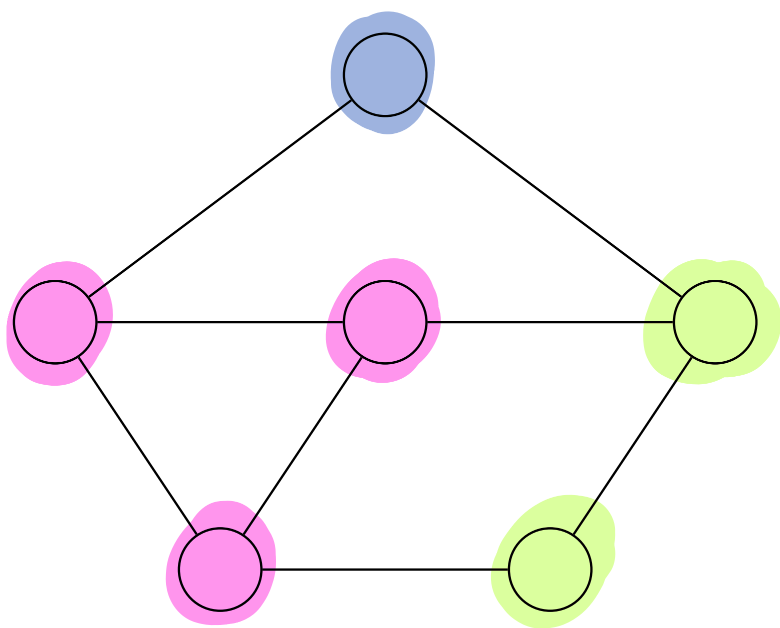
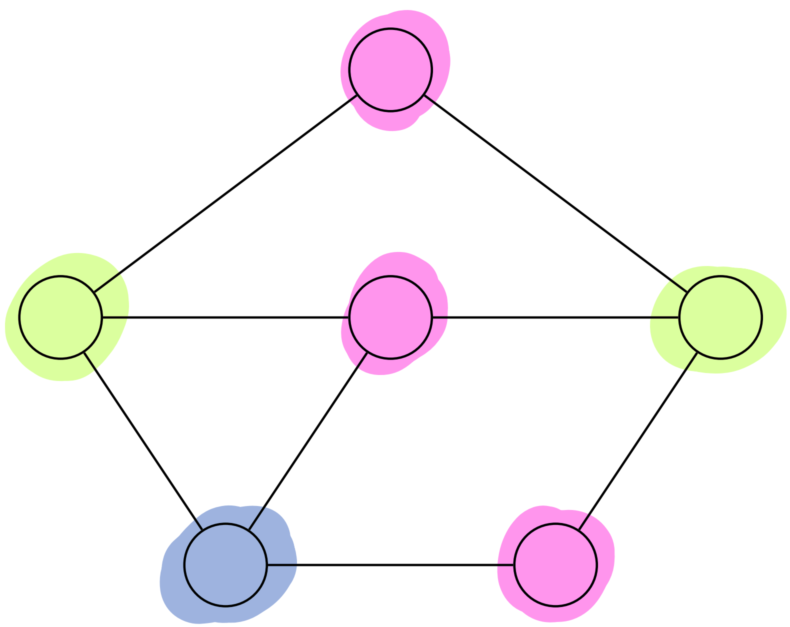


Cliques



Independent Sets

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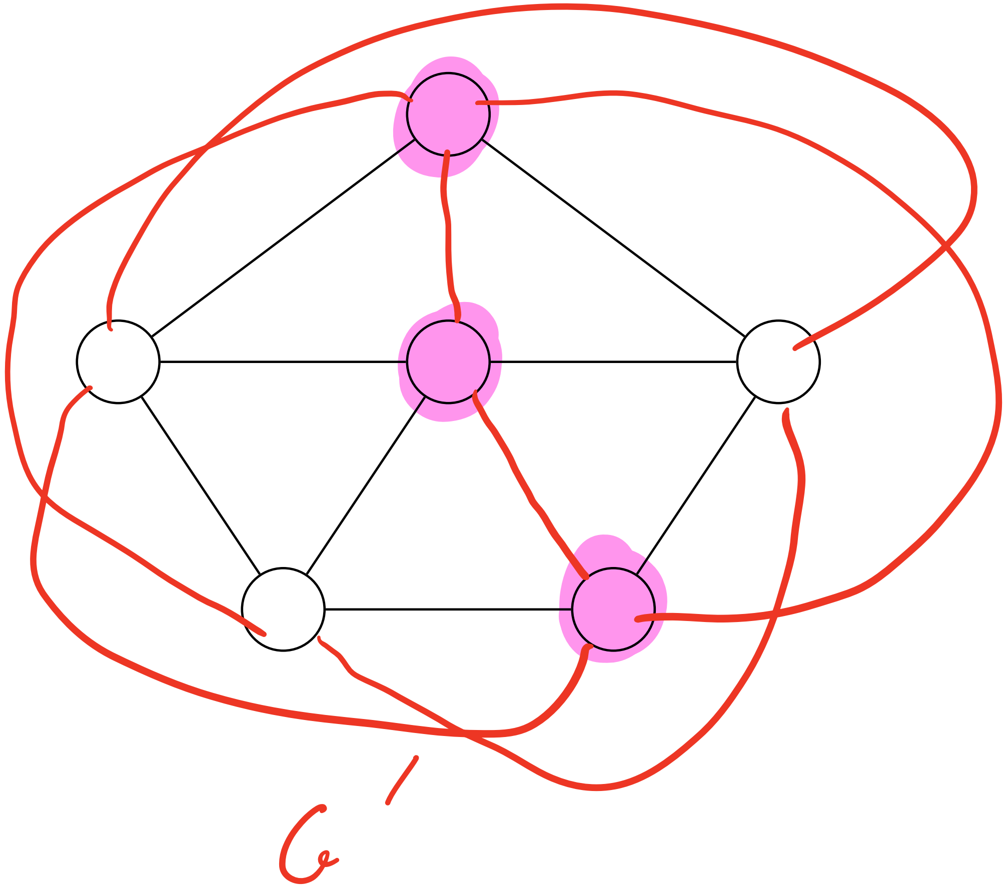
Problems of interest: given a graph G and an integer $1 \leq k \leq |V|$,

- Does G have a clique of size k?
- Does G have an independent set of size k?

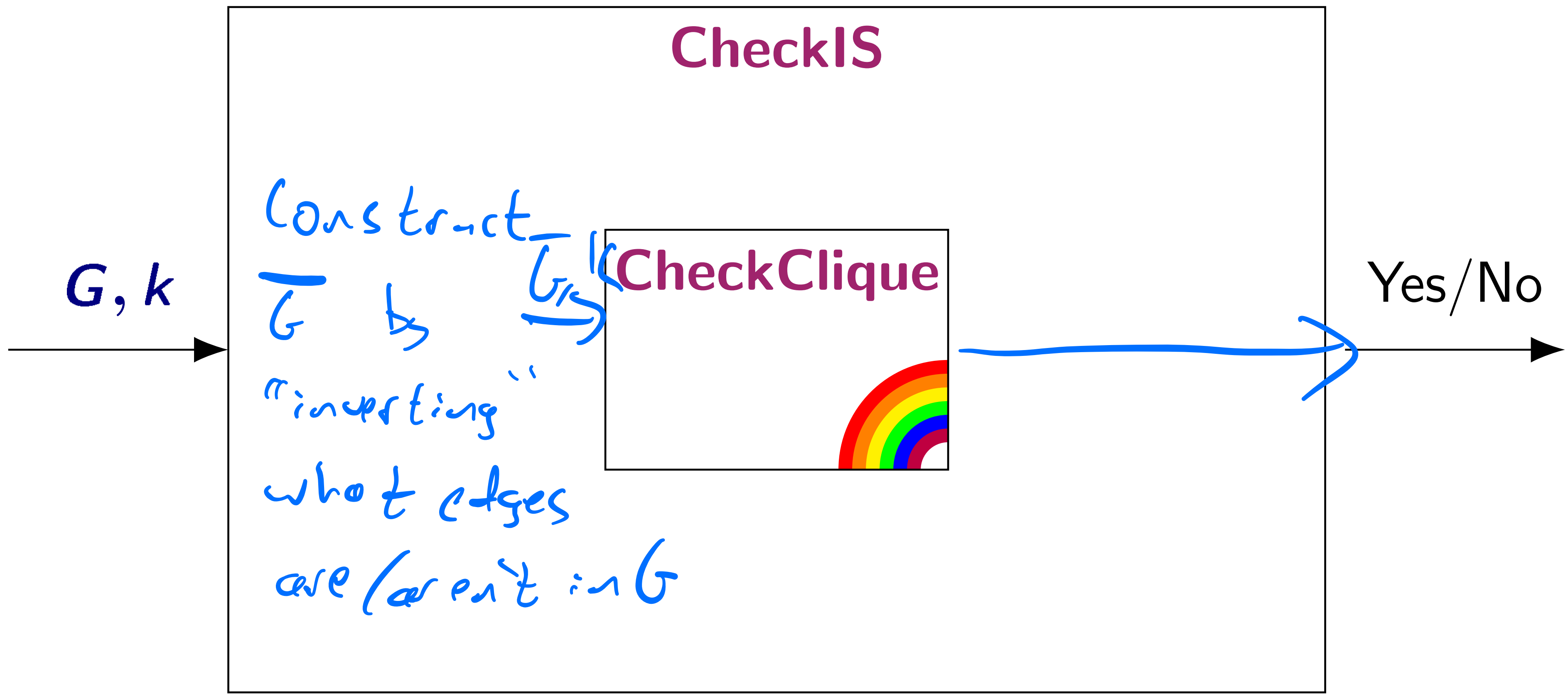
Reducing Independent Set to Clique I

Say we wanted to check if there is an independent set of size 3 in this graph. How can we use `CheckClique` to help?



Reducing Independent Set to Clique II

We want to check if G has an independent set of size k, given the ability to check if a graph has a clique of some size.



Reducing Clique to Independent Set

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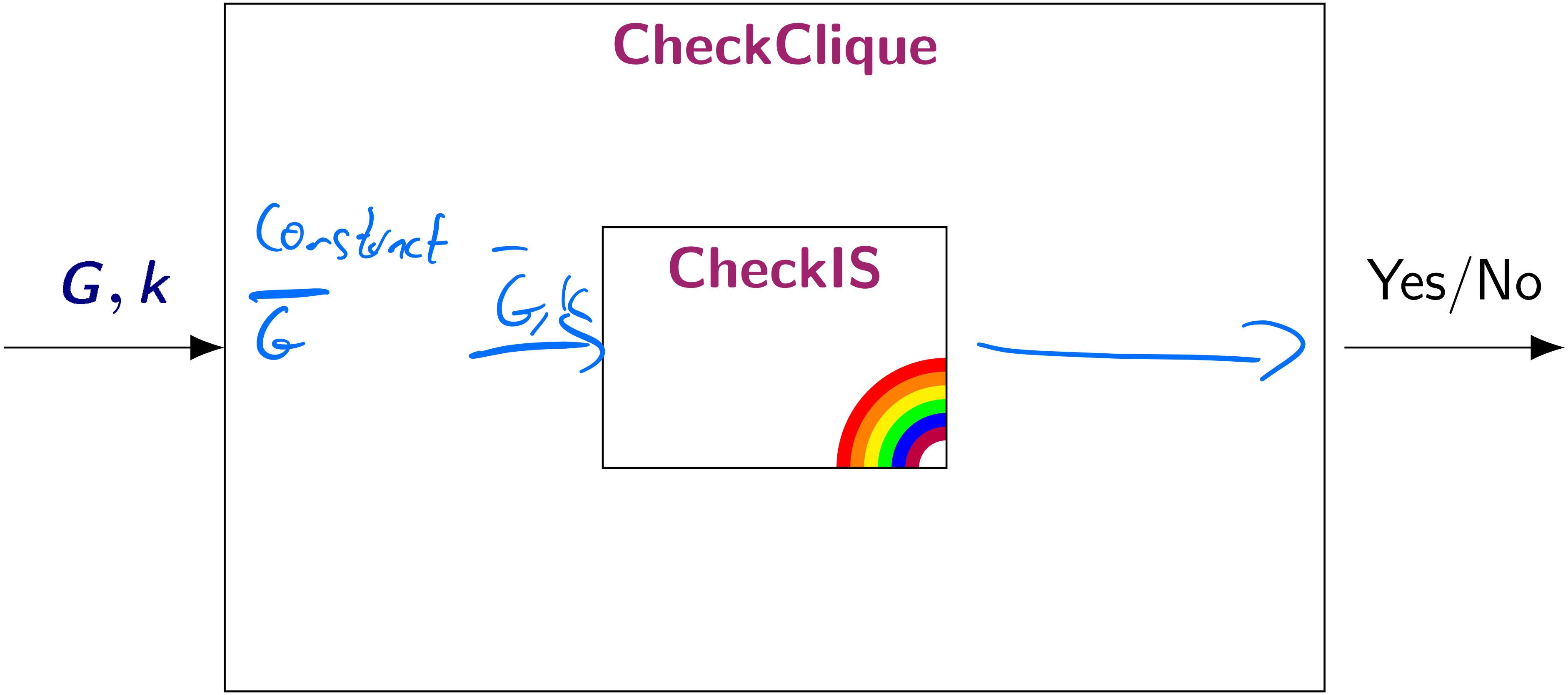

So Independent Set and Clique are "as easy / difficult" as each other!

Not All Reductions Work Both Ways

Both reductions we've seen so far can work in either direction—but this isn't always the case!

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Can we reduce checking for a clique to checking if a program halts?

```
from magic import TestHalt
CheckClique(G, k):
    construct program P:
      ① iterate over all C ⊆ V of size k
      ② if any is a valid clique, V is going to halt
         else, P enters an infinite loop
    return TestHalt(P)
```

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        ???
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It turns out this is impossible! (We'll see why in the next lecture.)

Takeaway: It matters which direction your reduction goes—some problems really are "strictly harder" than others!

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In practice, we often want to actually *find* an answer if one exists.

Concretely, how do the following problems compare?

- `CheckClique`(G, k): check if G has a clique of size k.
- `FindClique`(G, k): output a clique of size k if one exists, or say "Not possible".

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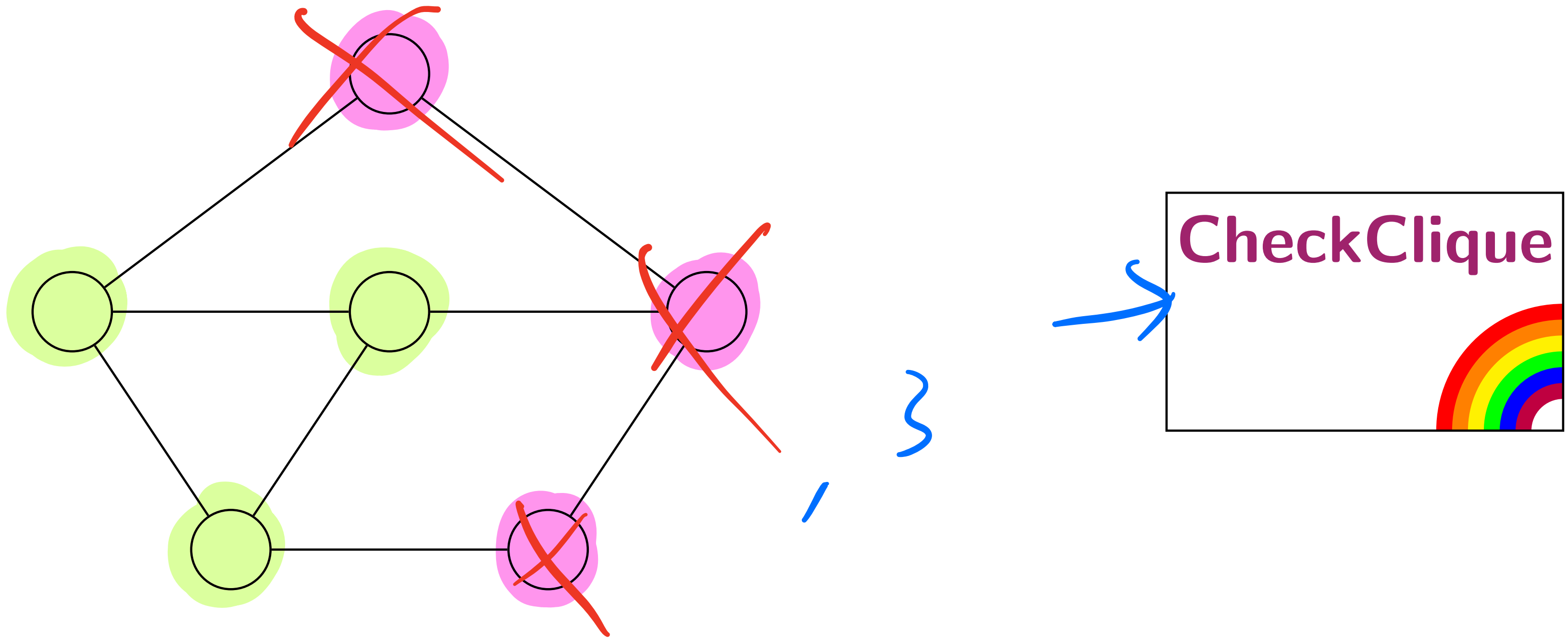
- `FindClique`(G, k): output a clique of size k if one exists, or say "Not possible".

Immediate: if we can solve `FindClique`, we can solve `CheckClique`.

Clique Search to Decision Intuition

Say we wanted to find a clique of size 3 in this graph. How can we use `CheckClique` to help?



Clique Search to Decision Reduction

```
from magic import CheckClique
FindClique(G, k):
    if CheckClique(G, k) is false:
        Return ``Not possible''
    for each vertex v ∈ G:
        Construct G' by removing v (and its edges) from G
        if CheckClique(G', k) is true:
            Remove v (and its edges) from G
    Return remaining vertices
```

Correctness?

1. at every step, we know G has a clique of size k
2. only leave v in if it is in every clique that remains

$\Rightarrow$ at the end every vertex is in every size k-clique

Clique Search to Decision Reduction

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from magic import CheckClique
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    if CheckClique(G, k) is false:
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        Construct G′ by removing v (and its edges) from G
        if CheckClique(G′, k) is true:
            Remove v (and its edges) from G
    Return remaining vertices
```

Correctness?

This means that checking if a clique exists and actually finding one “as easy / difficult” as each other!

Part III

Reductions for Decision Problems

Decision Problems

Similar to the first third of the class, we will be mostly interested in *decision* problems: our answer is either “Yes” or ”No”.

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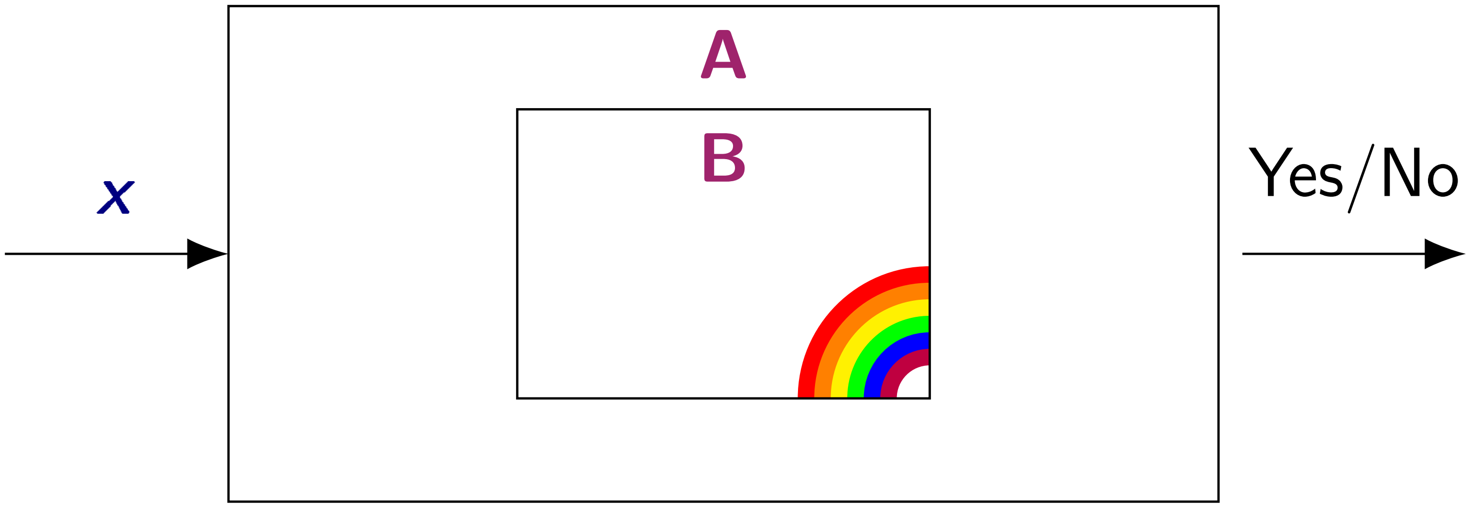
Recall that we often refer to such problems as a "language" $L \subseteq \{0,1\}^*$—strings in L are exactly those that we want to output "Yes" on (ie, accept).

Reducing Decision Problems

Say we have two decision problems $\boldsymbol{A}$ and $\boldsymbol{B}$. We can reduce $\boldsymbol{A}$ to $\boldsymbol{B}$ by giving a function $\boldsymbol{f}$ such that $\boldsymbol{x} \in \boldsymbol{L_A}$ if and only if $\boldsymbol{f(x)} \in \boldsymbol{L_B}$.



Reducing Decision Problems

Say we have two decision problems A and B. We can reduce A to B by giving a function f such that $x \in L_A$ if and only if $f(x) \in L_B$.

Examples from before:

- Reducing Independent Set to Clique, we took $f(G, k) = (\overline{G}, k)$.
- Reducing "Hello World!" to halting, we took $f(\texttt{StudentCode})$ to be the program P we defined.

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When using this type of reduction, you just have to define f and prove that $x \in L_A$ iff $f(x) \in L_B$!

- This will be the most common type of reduction we use because it is the most simple.

Takeaway Points

Reductions are a powerful tool in CS

- Reducing A to a problem with a known algorithm gives us an algorithm for A.
- Reducing a "hard" problem to B tells us that B must also be "hard".

To reduce A to B, write an algorithm for A where we can use a subroutine that solves B.

- *Don't* worry about how the subroutine for B is implemented—just use that it solves B!
- For decision problems, it suffices to give a function f such that $x \in L_A$ if and only if $f(x) \in L_B$.