Bellman-Ford and All-Pairs Shortest Paths

Lecture 19

April 3, 2025

Part I

SSSP with Negative Weights

Last Time

Want to solve the single-source shortest-path problem: given a vertex s, what is $\mathbf{dist}(s, t)$ for each vertex t?

- Weighted graphs: length is the sum of the weights of the edges.

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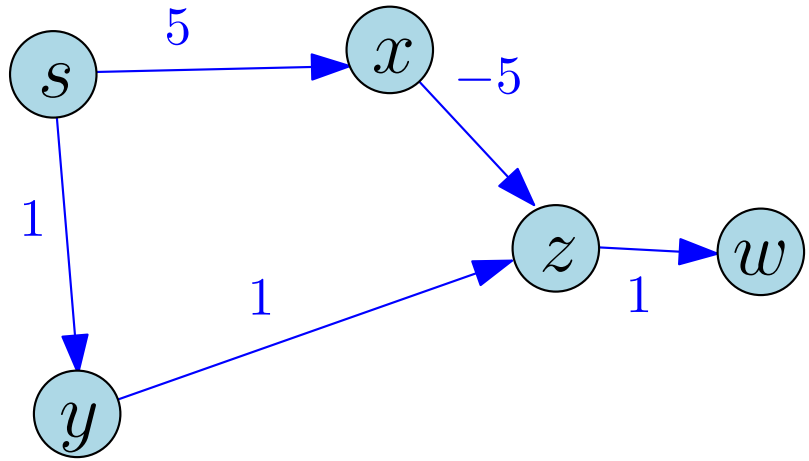
- Correct *as long as the edge weights are all non-negative*.
- May give the wrong answer if there are negative edge weights!



DP: Attempt 1

Why don't we use DP? (It worked for DAGs!)

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Subproblem definition:

Recurrence:

Evaluation Order:

DP: Attempt 2

How do we avoid circular dependencies? (Hint: add a subproblem parameter.)

Subproblem definition:

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DP Pseudocode

Let $\text{LSP}(v, \ell)$ be the length of the shortest path from s to v that uses at most ℓ edges.

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SSSP-DP(G, s):
    Initialize LSP as a V × V matrix
    Set LSP[s, 0] = 0 and LSP[v, 0] = ∞ for all v ≠ s
    for ℓ from 1 to V − 1:
        for all vertices v:
            LSP[v, ℓ] = LSP[v, ℓ − 1]
            for all edges (u, v):
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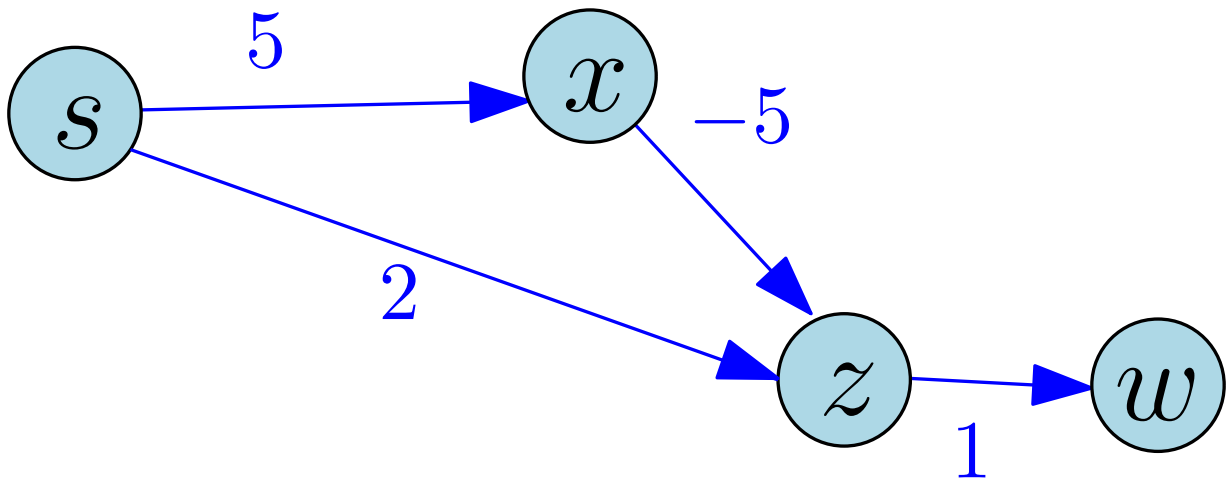
Efficiency?

DP Example



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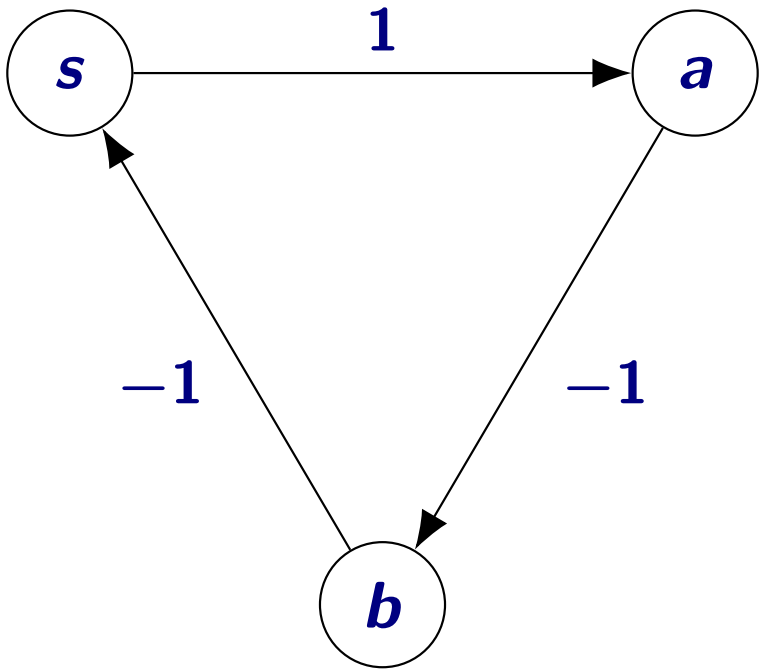
Correctness?
Proof by induction: after iteration i, $v.\text{dist} \leq \text{LSP}(v, i)$ for all v.

Just One Catch

What happens to Bellman-Ford on the following graph?



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(Technically, it's actually the concept of a shortest *walk* that breaks down. But note that all our algorithms implicitly are computing the shortest walk and using the fact that the shortest walk *must* be a path as long as there are no negative cycles. In fact, if we really want to find the shortest *path* with no restrictions on the weights at all, this turns out to be one of the canonically hard problems we study in the last part of the course!)

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Exercise: modify the pseudocode to output a negative cycle.

Negative Weights Takeaways

If G has no negative cycles, we can use `BellmanFord` to solve the SSSP problem in $O(VE)$ time.

- Can also *check* if G has a negative cycle!

If G may have negative cycles, the SSSP problem is (depending on your definitions) either (1) not well-defined, or (2) one of the canonical hard problems we'll consider in the last part of the class.

Part II

All-Pairs Shortest Paths

Recall APSP

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Naïve approach: run SSSP from each vertex.

- Non-negative weights (Dijkstra): $O(V(V + E) \log V)$ time
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Can we do better?

When In Doubt, Try Using DP

How would we write a DP algorithm for this?

Subproblem definition:

Recurrence:

Evaluation Order:

DP Pseudocode

Let $\text{LSP}(u, v, \ell)$ be the length of the shortest path from u to v that uses at most ℓ edges.

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APSP-DP(G):
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        for all vertices u:
            for all vertices v:
                LSP[u, v, ℓ] = LSP[u, v, ℓ − 1]
                for all edges (x, v):
                    LSP[u, v, ℓ] = min( LSP[u, v, ℓ],
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Efficiency?

Improving Our Approach

Can we come up with a more efficient recurrence?

Subproblem definition: Let $\mathrm{LSP}(u, v, \ell)$ be the length of the shortest path from u to v that uses at most ℓ edges.

Recurrence:

Evaluation Order:

Fischer-Meyer Pseudocode

Let $\text{LSP}(u, v, \ell)$ be the length of the shortest path from u to v that uses at most 2^ℓ edges.

```
FischerMeyer(G):
    Initialize LSP as a V × V × (⌈log₂(V)⌉ + 1) matrix
    Set LSP[u, v, 0] = w(u, v) for all u, v ∈ V
    # Let w(u, u) := 0, and for all (u, v) ∉ E let w(u, v) := ∞
    for ℓ from 1 to ⌈log₂(V)⌉:
        for all vertices u:
            for all vertices v:
                LSP[u, v, ℓ] = min_x (LSP[u, x, ℓ − 1] + LSP[x, v, ℓ − 1])
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Efficiency?

A Clever Subproblem

Subproblem definition:

Recurrence:

Evaluation Order:

Clever DP Pseudocode

Let $\text{LSP}(u, v, r)$ be the length of the shortest path from u to v *that only uses intermediate vertices from* $\{1, 2, \ldots r\}$.

```
APSP-CleverDP(G):
    Initialize LSP as a V × V × (V + 1) matrix
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    # Let w(u, u) := 0, and for all (u, v) ∉ E let w(u, v) := ∞
    for r from 1 to V:
        for all vertices u:
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        for all vertices u:
            for all vertices v:
                LSP[u, v, r] = min( LSP[u, v, r − 1],
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    return LSP[u, v, V] for all u, v ∈ V
```

Efficiency?

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FloydWarshall(G):
    Set dist$[u, v] = w(u, v)$ for all $u, v \in V$
    # Let $w(u, u) := 0$, and for all $(u, v) \notin E$ let $w(u, v) := \infty$
    for all vertices r:
        for all vertices u:
            for all vertices v:

$$\text{dist}[u, v] = \min \begin{pmatrix} \text{dist}[u, v], \\ \text{dist}[u, r] + \text{dist}[r, v] \end{pmatrix}$$

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    return $\texttt{dist}[u, v]$ for all $u, v \in V$

Correctness? Proof by induction: after iteration i of outermost loop, $\texttt{dist}[u, v] \leq \text{LSP}(u, v, i)$ for all $u, v \in V$.

Problems and Algorithms From Today

- Shortest paths from s with no negative cycles
  - Bellman-Ford, $O(VE)$
  - Can also check for negative cycles

- All-pairs shortest paths with no negative cycles
  - Negative weights: Floyd-Warshall, $O(V^3)$
  - Non-negative weights: Djikstra from each vertex, $O(V(V+E)\log V)$ (Floyd-Warshall is more efficient if $E \approx V^2$)