define vars z_{ij} = i^{th} bit of memory during step $\underline{j}$.

⋮

$\square$

Prop A If $L_1 \le_p L_2$ and $L_2 \in P$, then $L_1 \in P$.

Prop B If $L_1 \le_p L_2$ and $L_2 \le_p L_3$, then $L_1 \le_p L_3$.

Prop C Let L is an NP-complete problem.
then $L \notin P$ iff $P \ne NP$.

$(P \subseteq NP)$

Pf: $(\Rightarrow)$ Suppose $L \notin P$.
Then $L \in NP - P$, so $P \ne NP$.

$(\Leftarrow)$ Suppose $L \in P$.
Then $\forall L' \in NP$, $L' \le_p L$, $L \in P$
therefore, $L' \in P$.

$\therefore NP = P$. $\square$

Prop D If ① $L \in NP$ and
② $L_0 \le_p L$ for a known NP-complete problem L_0,
then L is NP-complete.

Pf: $\forall L' \in NP$, $L' \le_p L_0$ since L_0 is NPC
$L_0 \le_p L$
$\xRightarrow{\text{Prop B}}$ $L' \le_p L$. $\square$

3SAT

Input: Boolean formula F in 3CNF form (conjunctive normal)

3SAT

Input: Boolean formula F in $\underline{\text{3CNF}}$ form

$$\bigwedge_{i=1}^{m} (\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3})$$

where α_{ij} is a var or its complement (literal)

Output: yes iff $\exists$ assignment s.t. F evaluates to true

e.g. $F = \underbrace{(x_1 \vee x_2 \vee \overline{x_3})}_{\text{clause}} \wedge (\overline{x_1} \vee x_2 \vee x_4) \wedge (x_2 \vee x_3 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee \overline{x_4})$

(literal, literal, literal)

yes $(x_1 = 1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 0)$

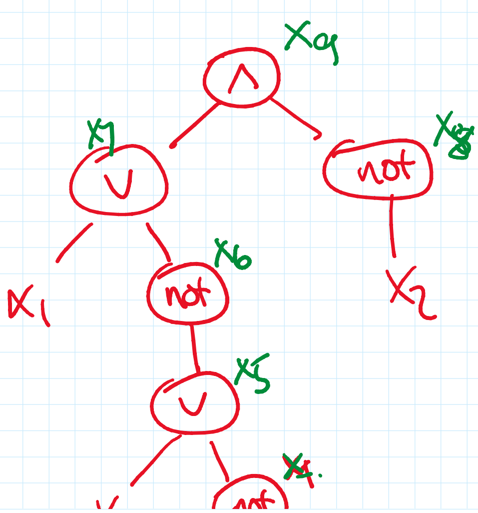
Thm 3SAT is NP-complete.

Pf Sketch: ① 3SAT $\in$ NP ✓

② Reduce from SAT to 3SAT: (SAT $\le_p$ 3SAT)

Given Boolean formula F, construct 3CNF Boolean formula F' as follows:

e.g. $F = (x_1 \vee \overline{x_2 \vee \overline{x_3}}) \wedge \overline{x_2}$



$(x_4 \leftrightarrow \overline{x_3})$

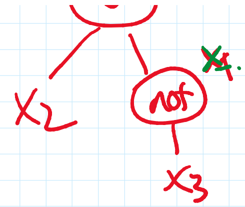
$\wedge\ (x_5 \leftrightarrow x_2 \vee x_4)$

$\wedge\ (x_6 \leftrightarrow \overline{x_5})$

$\wedge\ (x_7 \leftrightarrow x_1 \vee x_6)$

$\wedge\ (x_8 \leftrightarrow \overline{x_2})$

$\wedge\ (x_9 \leftrightarrow x_7 \wedge x_8)$



$\wedge (x_9 \Leftrightarrow x_7 \wedge x_8)$

$\wedge (x_9)$

⋮

"close" to 3CNF.

□

VERTEX-COV:

Input: undir graph $G=(V,E)$, int K

Output: yes iff $\exists$ vertex cov S of size $\leq K$

$\forall uv \in E \Rightarrow u \in S$ or $v \in S$

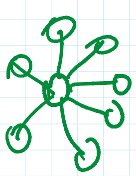
$u \notin S$ and $v \notin S \Rightarrow uv \notin E$

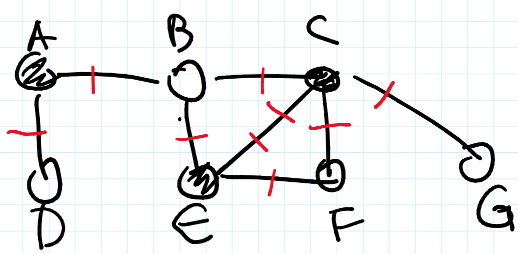
INDEP-SET: $p \to q$, $\bar{q} \to \bar{p}$

Input: G, K

Output: yes iff $\exists$ indep set S of size $\geq K$

$\forall u, v \in S \Rightarrow uv \notin E$.





min vertex cover $\{A, C, E\}$

max indep set $\{B, D, F, G\}$

CLIQUE: .. also equiv

Thm INDEP-SET is NP-complete.

Pf: ① INDEP-SET $\in$ NP ✓

② Reduction from 3SAT to INDEP-SET:

Given 3CNF formula F with n vars & m clauses

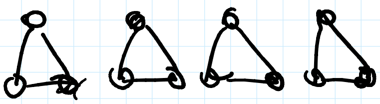
Given 3CNF formula F, with n vars & m clauses

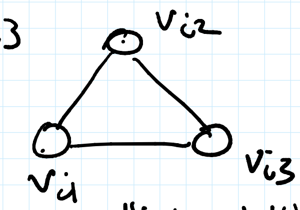
construct graph $\underline{G}$, and integer $\underline{K}$ as follows:

- for each clause $\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$,

  create 3 vertices v_{i1}, v_{i2}, v_{i3}

  & 3 edges $v_{i1}v_{i2}$, $v_{i2}v_{i3}$, $v_{i3}v_{i1}$





(a "triangle")

- whenever $\alpha_{ij} = \overline{\alpha_{i'j'}}$,

  add an edge $v_{ij} v_{i'j'}$

  "cross" edge

- let $K = m$.

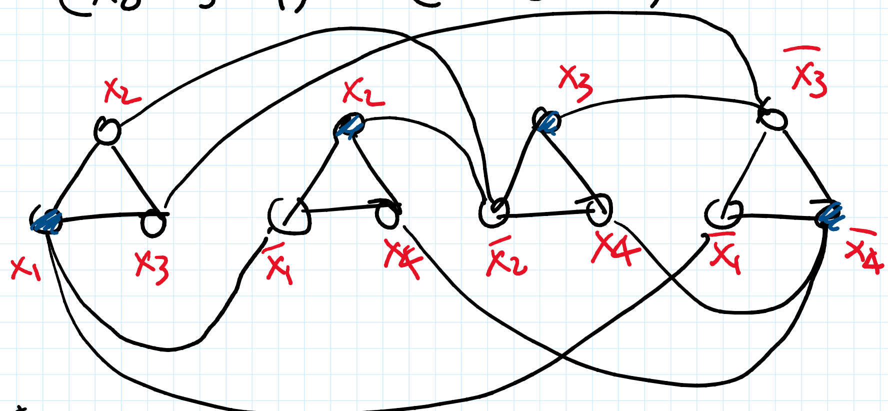
$O(m^2)$

This construction from F to (G, K) takes polytime.

Ex given $F = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4) \wedge (\overline{x_2} \vee x_3 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_3} \vee \overline{x_4})$

$x_1 = x_2 = x_3 = 1$
$x_4 = 0$

construct G:



$K = 4$

Correctness: $\exists$ assignment that makes F true $\iff$ $\exists$ indep set for G of size K

Pf of ($\Rightarrow$): Given sat assignment,

define a subset S as follows:

for each clause $\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$

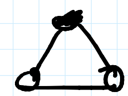
pick any j s.t. α_{ij} true

put v_{ij} in S.

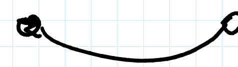
Then $|S| = m = K$

& S is an indep set

check triangle edges



check cross edges. ✓



Pf of ($\Leftarrow$):

$\vdots$