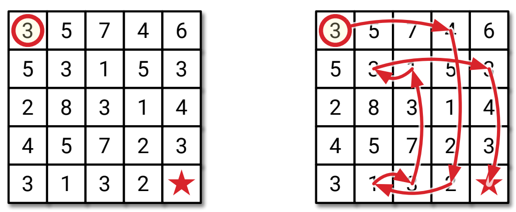
For each of the problems below, transform the input into a graph and apply a standard graph algorithm that you've seen in class. Whenever you use a standard graph algorithm, you ***must*** provide the following information. (I recommend actually using a list as the one used below.)

- What are the vertices?
- What are the edges? Are they directed or undirected?
- If the vertices and/or edges have associated values, what are they?
- What problem do you need to solve on this graph?
- What standard algorithm are you using to solve that problem?
- What is the running time of your entire algorithm, *including* the time to build the graph, *as a function of the original input parameters*?

1 A ***number maze*** is an $n \times n$ grid of positive integers. A token starts in the upper left corner; your goal is to move the token to the lower-right corner. On each turn, you are allowed to move the token up, down, left, or right; the distance you may move the token is determined by the number on its current square. For example, if the token is on a square labeled 3, then you may move the token three steps up, three steps down, three steps left, or three steps right. However, you are never allowed to move the token off the edge of the board.



A 5×5 number maze that can be solved in eight moves.

Describe and analyze an efficient algorithm that either returns the minimum number of moves required to solve a given number maze $M[1 \,..\, n, 1 \,..\, n]$, or correctly reports that the maze has no solution.

Solution:

We construct a directed graph $G = (V, E)$ as follows.

- The vertices of G correspond to (indices of) grid cells:

$$V := \{i \mid 1 \le i \le n\} \times \{j \mid 1 \le i \le n\}$$

  There are exactly n^2 vertices.

- The directed edges of G correspond to legal moves; more formally, $E = E_{\leftrightarrow} \cup E_{\updownarrow}$, where

$$\begin{aligned} E_{\leftrightarrow} &:= \big\{(i,j)\to(i,j') \;\big|\; |j-j'| = M[i,j]\big\} && \text{[horizontal edges]} \\ E_{\updownarrow} &:= \big\{(i,j)\to(i',j) \;\big|\; |i-i'| = M[i,j]\big\} && \text{[vertical edges]} \end{aligned}$$

  There are at most $4n^2$ edges.

- The vertices and edges do not have associated values. In particular, each integer $M[i,j]$ is already encoded into the edges leaving vertex (i,j).

- Any sequence of legal moves in number maze M corresponds to a directed path in graph G, and vice versa. In particular, the shortest sequence of legal moves from the upper left corner to the lower right corner corresponds to the shortest path in G from $(1,1)$ to (n,n).
- We can compute this shortest path using breadth-first search.
- Our algorithm runs in $O(V+E) = \boldsymbol{O(n^2)}$ ***time***.

The remaining problems consider variants of problem 1, where the sequence of moves must satisfy certain constraints to be considered a valid solution. For each problem, your goal is to describe and analyze an algorithm that either returns the minimum number of moves in a *valid* solution to a given number maze, or reports correctly that no valid solution exists.

2 Suppose a sequence of moves is considered *valid* if and only if the moves alternate between horizontal and vertical. That is, a valid move sequence never has two horizontal moves in a row or two vertical moves in a row.

Describe and analyze an algorithm that either returns the minimum number of moves in a valid solution to a given number maze, or reports correctly that no valid solution exists.

Solution:

We construct a directed graph $G = (V, E)$ as follows.

- There are two vertices of G for each grid cell, each corresponding to the orientation of the next move out of that cell.

$$V := \{i \mid 1 \le i \le n\} \times \{j \mid 1 \le i \le n\} \times \{\updownarrow, \leftrightarrow\}$$

  There are exactly $2n^2$ vertices.

- The directed edges of G correspond to legal moves; more formally, $E = E_{\leftrightarrow} \cup E_{\updownarrow}$, where

$$\begin{aligned} E_{\leftrightarrow} &:= \big\{(i,j,\leftrightarrow)\to(i,j',\updownarrow) \bigm| \left|j-j'\right| = M[i,j]\big\} && \text{[horizontal edges]} \\ E_{\updownarrow} &:= \big\{(i,j,\updownarrow)\to(i',j,\leftrightarrow) \bigm| \left|i-i'\right| = M[i,j]\big\} && \text{[vertical edges]} \end{aligned}$$

  There are at most $4n^2$ edges.

- The vertices and edges do not have associated values. In particular, each integer $M[i,j]$ is already encoded into the edges leaving vertex (i,j).
- Any sequence of legal moves in number maze M corresponds to a directed path in graph G, and vice versa. In particular, the shortest sequence of legal moves from the upper left corner to the lower right corner corresponds to the shortest path in G from either $(1,1,\updownarrow)$ or $(1,1,\leftrightarrow)$, to either $(n,n,\updownarrow)$ or $(n,n,\leftrightarrow)$.
- We can compute all four shortest paths from $(1,1,?)$ to $(n,n,?)$ using two calls two breadth-first search, starting at $(1,1,\updownarrow)$ and $(1,1,\leftrightarrow)$, and then return the shortest of those four shortest paths.
- Our algorithm runs in $O(V+E) = \boldsymbol{O(n^2)}$ ***time***.

The graph G is a standard product construction between the basic number-maze graph, described in our solution to problem 1, and a cycle of length 2.

3 Suppose a sequence of moves is considered *valid* if and only if its length (the number of moves) is a multiple of 5.

Describe and analyze an algorithm that either returns the minimum number of moves in a valid solution to a given number maze, or reports correctly that no valid solution exists.

Solution:

We construct a directed graph $G = (V, E)$ as follows.

- There are five vertices of G for each grid cell, each corresponding to a different remainder mod 5:

$$V := \{i \mid 1 \le i \le n\} \times \{j \mid 1 \le i \le n\} \times \{0, 1, 2, 3, 4\}$$

  There are exactly $5n^2$ vertices.

- The directed edges of G correspond to legal moves; more formally, $E = E_{\leftrightarrow} \cup E_{\updownarrow}$, where

$$E_{\leftrightarrow} := \big\{(i, j, r) \to (i, j', r + 1 \bmod 5) \mid |j - j'| = M[i, j]\big\}$$
$$E_{\updownarrow} := \big\{(i, j, r) \to (i', j, r + 1 \bmod 5) \mid |i - i'| = M[i, j]\big\}$$

  There are at most $20n^2$ edges.

- The vertices and edges do not have associated values. In particular, each integer $M[i, j]$ is already encoded into the edges leaving each vertex (i, j, r).
- Any valid sequence of legal moves in number maze M corresponds to a directed path in graph G, and vice versa. In particular, the shortest valid sequence of legal moves from the upper left corner to the lower right corner corresponds to the shortest path in G from $(1, 1, 0)$ to $(n, n, 0)$.
- We can compute this shortest path using breadth-first search.
- Our algorithm runs in $O(V + E) = \boldsymbol{O(n^2)}$ ***time***.

The graph G is a standard product construction between the basic number-maze graph, described in our solution to problem 1, and a cycle of length 5.

Harder problems to think about later:

4 Now suppose a sequence of moves is considered *valid* if and only if it contains no U-turns.

Describe and analyze an algorithm that either returns the minimum number of moves in a valid solution to a given number maze, or reports correctly that no valid solution exists.

Solution:

The main idea is to keep track not only of the position of the token, but the direction of its last move.

First let $G = (V, E)$ be the directed graph of grid squares and legal moves from problem 1. We construct a new directed graph $G' = (V', E')$ as follows:

- The vertices V' correspond to *legal moves* in the number maze, except for a special start vertex indicating that no moves have yet been made. More formally, we have

$$V' = \{(1, 1)\} \cup E.$$

  Each vertex $u \to v$ corresponds to the last move taken by the token; vertex $(1, 1)$ corresponds to the token's start in the upper left corner. There are at most $4n^2 + 1$ vertices in V'.

- The directed edges E' correspond to legal initial moves and valid consecutive pairs of legal moves:
  - $(1,1) \Rightarrow ((1,1) \rightarrow v)$ for every legal move $(1,1) \rightarrow v \in E$
  - $(u \rightarrow v) \Rightarrow (v \rightarrow w)$ for each pair of legal moves $u \rightarrow v \in E$ and $v \rightarrow w \in E$ that share a grid cell v and do not point in opposite directions.

  (I'm using double arrows to denote directed edges in E', to distinguish them from edges in E.) There are at most 2 legal moves from the starting cell, and for any legal move there are at most three valid next moves. Thus, there are at most $2 + 12n^2$ edges in E'.
- The vertices and edges do not have associated values.
- Any valid sequence of ℓ legal moves in M corresponds to a directed path in G' with length ℓ. For example, the 4-move sequence $(1,1) \rightarrow (3,1) \rightarrow (3,7) \rightarrow (4,7) \rightarrow (4,4)$ corresponds to the directed path

$$(1,1) \Rightarrow ((1,1) \rightarrow (3,1)) \Rightarrow ((3,1) \rightarrow (3,7)) \Rightarrow ((3,7) \rightarrow (4,7)) \Rightarrow ((4,7) \rightarrow (4,4)),$$

  which has length 4. Thus, the shortest valid sequence of legal moves from $(1,1)$ to (n,n) corresponds to the shortest path in G from the start vertex $(1,1)$ to any vertex of the form $u \rightarrow (n,n)$.
- We can compute this shortest path using breadth-first search. Specifically, we compute shortest path in G' from $(1,1)$ to every other vertex in V' using breadth-first search, and then find the best last move $u \rightarrow (n,n)$ by comparing distances by brute force.
- Our algorithm runs in $O(V + E) = \boldsymbol{O(n^2)}$ ***time***.

5 Now suppose a sequence of moves is considered *valid* if and only if the sequence of move *lengths* is increasing. Describe and analyze an algorithm that either returns the minimum number of moves in a valid solution to a given number maze, or reports correctly that no valid solution exists.

Solution:

We construct a directed graph $G = (V, E)$ as follows.

- The vertices of G correspond to (indices of) grid cells and the length of the move that brought the token to that grid cell:

$$V := \{i \mid 1 \le i \le n\} \times \{j \mid 1 \le i \le n\} \times \{\ell \mid 0 \le i \le n-1\}$$

  There are exactly n^3 vertices.
- The directed edges of G correspond to legal moves; more formally, $E = E_{\leftrightarrow} \cup E_{\updownarrow}$, where

$$\begin{aligned} E_{\leftrightarrow} &:= \big\{(i,j,\ell) \rightarrow (i,j',|j-j'|) \;\big|\; |j-j'| = M[i,j] > \ell\big\} && \text{[horizontal edges]} \\ E_{\updownarrow} &:= \big\{(i,j,\ell) \rightarrow (i',j,|i-i'|) \;\big|\; |i-i'| = M[i,j] > \ell\big\} && \text{[vertical edges]} \end{aligned}$$

  There are at most $4n^3$ edges.
- The vertices and edges do not have associated values.
- Any sequence of legal moves in number maze M corresponds to a directed path in graph G, and vice versa. In particular, the shortest sequence of legal moves from the upper left corner to the lower right corner corresponds to the shortest path in G from the start vertex $(1,1,0)$ to any vertex of the form (n,n,ℓ).
- We can compute shortest paths from $(1,1,0)$ to every vertex in V using breadth-first search, after which we can find the shortest shortest path to $(n,n,\cdot)$ by brute force.

- Our algorithm runs in $O(V+E) = \boldsymbol{O(n^3)}$ ***time***.

This graph G is a *subgraph* of the standard product construction between the basic number-maze graph, described in our solution to problem 1, and a complete directed acyclic graph with n vertices.

6 Finally, suppose a sequence of moves is considered *valid* if and only if the sequence of move lengths is a palindrome. (A palindrome is any sequence that is equal to its reversal.)

Describe and analyze an algorithm that either returns the minimum number of moves in a valid solution to a given number maze, or reports correctly that no valid solution exists.

Solution:

The main idea is to keep track of both ends of the move sequence and move inward. Let $G = (V, E)$ be the directed graph constructed in our solution ot problem 1. We construct a new directed graph $G' = (V', E')$ as follows.

- The vertices of G' correspond to *pairs* of vertices in G

$$V' := V \times V = \big\{(i,j;k,l) \bigm| 1 \le i,j,k,l \le n\big\}$$

  There are exactly n^4 vertices.

- The directed edges of G correspond to pairs of legal moves, forward from the first cell and backward from the second cell, with the same length

$$E' = \left\{ (i,j;k,l) \!\rightarrow\! (i',j';k',l') \;\middle|\; \begin{array}{c} (i,j)\!\rightarrow\!(i',j') \in E \\ (k',l')\!\rightarrow\!(k,l) \in E \\ M[i,j] = M[k',l'] \end{array} \right\}$$

  There are at most $16n^4$ edges.

- The vertices and edges do not have associated values/
- Each *palindromic* sequence of legal moves in number maze M corresponds to a directed path in graph G, and vice versa. In particular, the shortest palindromic sequence of legal moves from the upper left corner to the lower right corner corresponds to the shortest path in G from $(1,1;n,n)$ to any vertex of the form $(i,j;i,j)$ (for even-length palindromes) or $(i,j;k,l)$ where $(i,j)\!\rightarrow\!(k,l) \in E$ (for odd-length palindromes).
- We can compute shortest paths from $(1,1;n,n)$ to every vertex in V' using breadth-first search, after which we can find the shortest relevant shortest path by brute force comparison.
- Our algorithm runs in $O(V+E) = \boldsymbol{O(n^4)}$ ***time***.

The graph G' is a standard product construction between the basic number-maze graph G and its reversal.