Regular Languages and Expressions

Lecture 2
January 19, 2023

Background

Fix some finite alphabet Σ.

- Σ^* is the set of all strings over Σ
- A language over Σ is a subset of strings. That is, $L \subseteq \Sigma^*$
- Σ^* is countably infinite. Set of all languages $= \mathcal{P}(\Sigma^*)$ is uncountably infinite
- Each machine/program can be described by a string. Hence set of machines/programs is countably infinite
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Question: What languages are easy? What languages should we focus on? Can we *classify* them via various features?

Languages

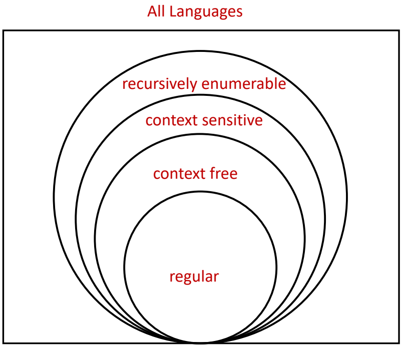
Study of languages motivated by (among many others)

- linguistics and natural language understanding
- programming languages and logic
- computation and machines

Intution: As ability of a language to *express/model* increases the more *complex/computationally hard* it becomes.

Chomsky Hierarchy and Machines

Grammars	Machines
phrase structured	Turing machine (TMss)
context sensitive	linear bounded automata (LBAs)
context free	pushdown automata (PDAs)
regular expressions	finite state automata (DFAs)



Part I

Regular Languages

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A class of simple but very useful languages.

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Regular languages are closed under the operations of union, concatenation and Kleene star.

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If w is a string then $L = \{w\}$ is regular.

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Lemma

Every finite language L is regular.

Examples: $L = \{a, abaab, aba\}$. $L = \{w \mid |w| \leq 100\}$. Why?

More Examples

- $\{w \mid w$ is a keyword in Python program$\}$
- $\{w \mid w$ is a valid date of the form mm/dd/yy$\}$
- $\{w \mid w$ describes a valid Roman numeral$\}$ $\{I, II, III, IV, V, VI, VII, VIII, IX, X, XI, \ldots\}$.
- $\{w \mid w$ contains "CS374" as a substring$\}$.

Regular Languages

- How expressive are these languages?
- What can we use them for?
- What are limitations? That is, what can be *not* express as regular languages?

Part II

Regular Expressions

Regular Expressions

A way to denote regular languages

- simple patterns to describe related strings
- useful in
  - text search (editors, Unix/grep, emacs)
  - compilers: lexical analysis
  - compact way to represent interesting/useful languages
  - dates back to 50's: Stephen Kleene
    who has a star named after him.

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Base cases:

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Inductive cases: If r_1 and r_2 are regular expressions denoting languages R_1 and R_2 respectively then,

- $(r_1 + r_2)$ denotes the language $R_1 \cup R_2$
- $(r_1 r_2)$ denotes the language $R_1 R_2$
- $(r_1)^*$ denotes the language R_1^*

Regular Languages vs Regular Expressions

Regular Languages	Regular Expressions
$\emptyset$ regular	$\emptyset$ denotes $\emptyset$
$\{\epsilon\}$ regular	ϵ denotes $\{\epsilon\}$
$\{a\}$ regular for $a \in \Sigma$	a denote $\{a\}$
$R_1 \cup R_2$ regular if both are	$\mathrm{r}_1 + \mathrm{r}_2$ denotes $R_1 \cup R_2$
$R_1 R_2$ regular if both are	$\mathrm{r}_1\mathrm{r}_2$ denotes $R_1 R_2$
R^* is regular if R is	r^* denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

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Examples: $(0 + 1)^*$, $010^* + (110)^*$, $(10 + 110)^*(11 + 10)$

Notation and Parenthesis

- For a regular expression r, $\boldsymbol{L}(\mathrm{r})$ is the language denoted by r. Multiple regular expressions can denote the same language!
  Example: $(0 + 1)$ and $(1 + 0)$ denote same language $\{0, 1\}$

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- Other notation: $r + s$, $r \cup s$, $r|s$ all denote union. rs is sometimes written as $r \bullet s$.

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- Given a regular expression r we would like to "understand" $L(r)$ (say by giving an English description)

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- Hard: bitstrings with an odd number of 1s *and* an odd number of 0s.

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Question: How does on prove an identity?
By induction. On what? Length of r since r is a string obtained from specific inductive rules.

A non-regular language and other closure properties

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Other questions:

- Suppose R_1 is regular and R_2 is regular. Is $R_1 \cap R_2$ regular?
- Suppose R_1 is regular is $\bar{R}_1$ (complement of R_1) regular?

Summary and Skills

Regular languages and expressions defined inductively via simple base cases and three operations: union, concatenation, Kleene star

Skills:

- Given a laguage L described in English, design a regular expression r such that $L = L(r)$
- Given a regular expression r, give an English description of the language $L(r)$

Later:

- see equivalence with DFAs, NFAs
- technique to prove that languages are not regular