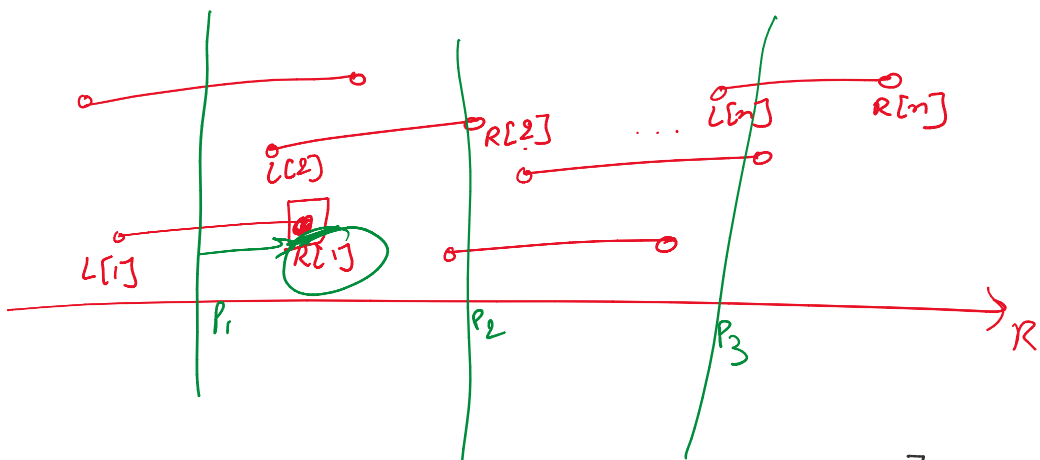
Tuesday, April 30, 2024 9:13 AM

Please Fill out the ICES forms if you haven't already: https://ices.citl.illinois.edu/

Final Exam: Cummulative. Look up SP'22 Final for "high-level" format.

- Greedy
- NPC.

38 Let X be a set of n intervals on the real line. We say that a set P of points *stabs* X if every interval in X contains at least one point in P. Describe and analyze an efficient algorithm to compute the smallest set of points that stabs X. Assume that your input consists of two arrays $L[1..n]$ and $R[1..n]$, representing the left and right endpoints of the intervals in X. If you use a greedy algorithm, don't forget to ***prove*** that it is correct.



Input: n intervals. i^{th} interval is $[L[i], R[i]]$

O/P: P of points s.t. $\forall i \in N$, $\exists p \in P$ s.t. $L[i] \le p \le R[i]$

minimizing $|P|$.

Greedy Strat: Pick earliest end-point as the first p in P $\equiv p = \arg\min_i R[i]$

★ Alg'm :

$O(n\log n) \leftarrow$ 1. Sort $R[1] \le R[2] \le \dots \le R[n]$

2. $\mathcal{P} = \phi$, $\underline{RI}$ = set of all i/p intervals.

3. while $RI \neq \phi$ {

4. p = earliest end-point among the intervals in RI.

5. $\mathcal{P} = \mathcal{P} \cup \{p\}$, $\boxed{RI = RI \setminus \text{All intervals stabbed by } p}$

$= \{ i \in RI \mid L[i] \le p \le R[i] \}$

}

$O(n)$ exe. (sort L as well)

6. Return $\mathcal{P}$.

★ Correctness Pf : (By Induction on n = # intervals)

$\boxed{\text{Recall: } R[1] \le R[2] \le \dots \le R[n]}$, $R[1] = p$

★ Base Case : $n=1$. Then $\mathcal{P} = \{R[1]\}$ is OPT.

★ Inductive Hypothesis : For any instance I' w/ $n' < n$ intervals, suppose Greedy o/p is OPT.

★ Induction : Given instance I w/ n intervals.

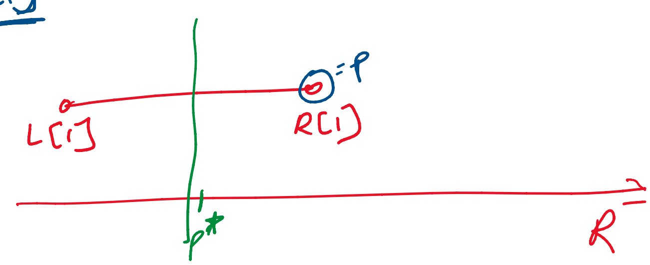
Let,

OPT : optimum sol'n (unknown). p^* : left-most point in OPT $\left(p^* = \min_{q \in OPT} q\right)$

$\mathcal{P}$: o/p of Greedy Alg'm. p : left-most point in $\mathcal{P}$ $\left(p = \min_{q \in \mathcal{P}} q\right)$

We know, $p = R[1] \le \underline{R[i]}$

& $\boxed{p^* \le p.}$



Claim: If $[L[i], R[i]]$ is stabbed by p^* then it is also stabbed by p.

Claim: If $[L[i], R[i]]$ ... it is also stabbed by p.

For $[L[i], R[i]]$ if $\boxed{L[i] \le p^*} \le R[i]$ then
$L[i] \le p \le R[i]$.

pf: $L[i] \le p^* \le p \le R[1] \le R[i]$ $\quad (\because p^* \le p,\ R[1] \le R[i])$ ∎

The claim implies that the set of intervals stabbed by p^*
$\subseteq$ set of intervals " " p.

(exchange arg.)

$\widetilde{OPT} = OPT \setminus \{p^*\} \cup \{p\}$. Then by the above claim
$\widetilde{OPT}$ is also an optimum sol'n.

Let, $I' = I \setminus \{\text{intervals stabbed by } p = R[1]\}$

Clearly, $|I'| \le (n-1) < n$.

$P \setminus \{p\}$ is a greedy sol'n for I'

AND $P \setminus \{p\}$ is an OPT " " I'

$\Rightarrow$ $P \setminus \{p\}$ is an OPT

($\because$ I.H.)

$\widetilde{OPT} \setminus \{p\}$ is also an OPT sol'n for I'

Since, $\widetilde{OPT} \setminus \{p\}$ is also an OPT sol'n for I'

it must be that $|P \setminus \{p\}| = |\widetilde{OPT} \setminus \{p\}|$

$\Rightarrow$ P is an OPT sol'n for I. ∎

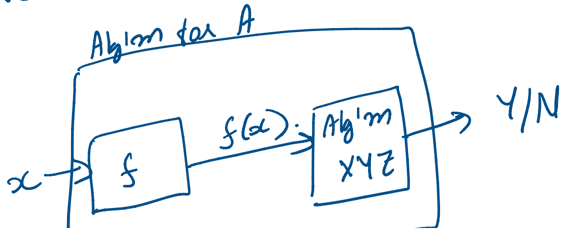
---

o/p.

✪ NP - Hardness.

How to show that problem XYZ is NP-hard?

Intuition: Take an NP-hard problem, say A (= 3SAT)



correctness of Alg'm for A:

claim 1: YES for i/p x of A $\Rightarrow$ YES for i/p $f(x)$ of XYZ

claim 2: $\neg$(NO for i/p x of A $\Rightarrow$ NO for i/p $f(x)$ of XYZ)

$\equiv$ YES for i/p $f(x)$ of XYZ $\Rightarrow$ YES for i/p x of A)

claims 1 & 2 $\equiv$ YES for i/p x of A $\Leftrightarrow$ YES for i/p $f(x)$ of XYZ

A: known NP-hard problem.

Show that $A \le_p$ XYZ Independent set (IS)

eg. 3SAT $\le_p$

in poly-time

★ step 1: Given i/p x of A $\longrightarrow$ Construct i/p $f(x)$ of XYZ

Given 3CNF formula ϕ $\longrightarrow$ construct graph G, number k. (G, k)

★ step 2: Show that $\exists$ sol'n for x of A $\Leftrightarrow$ $\exists$ sol'n for $f(x)$ of XYZ

$\exists$ satisfying assignment for ϕ $\Leftrightarrow$ $\exists$ IS of size $\ge k$ in G

($\Rightarrow$) Given a sol'n for x $\rightarrow$ construct sol'n for $f(x)$

Given a satisfying assignment for ϕ $\rightarrow$ construct IS of size $\ge k$ in G.

($\Leftarrow$) Given a sol'n for $f(x)$ $\rightarrow$ construct sol'n for x

Given an IS of size $\ge k$ in G $\rightarrow$ construct satisfying assignment for ϕ

★ Now to show VC (vertex cover) is NP-hard, we should

VC $\le_p$ IS? OR

$\boxed{\text{IS} \le_p \text{VC?}}$

31 To celebrate the end of the semester, Professor Jarling wants to treat himself to an ice-cream cone at the *Polynomial House of Flavors*. For a fixed price, he can build a cone with as many scoops as he'd like. Because he has good balance (and because we want this problem to work out), Prof. Jarling can balance any number of scoops on top of the cone without it tipping over. He plans to eat the ice cream one scoop at a time, from top to bottom, and doesn't want more than one scoop of any flavor.

However, he realizes that eating a scoop of bubblegum ice cream immediately after the scoop of potatoes-and-gravy ice cream would be unpalatable; these two flavors clearly should not be placed next to each other in the stack. He has other similar constraints; certain pairs of flavors cannot be adjacent in the stack.

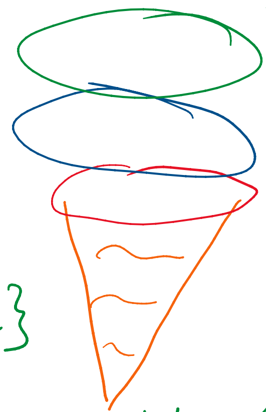
He'd like to get as much ice cream as he can for the one fee by building the tallest cone possible that meets his flavor-incompatibility constraints. Prove that Prof. Jarling's problem is NP-hard.

Given: set of flavours

his flavor-incompatibility constraints. Prove that Prof. Jarling's problem is NP-hard.

Given: set of flavours $\{f_1, \dots, f_n\}$

Constraints $C = \{(f_i, f_j) \mid f_i \ \& \ f_j$ can not be placed next to each other$\}$



o/p: max # scoops, at most one per flavour, that professor can have.

known NP-hard problems: IS / HC / HP / DHC / DHP / 3-coloring / 3-SAT / VC / SC

★ step-1: Hamiltonian-Path $\le_p$ Ice-Cream-Decision.

i/p: Undirected graph $G = (V, E)$

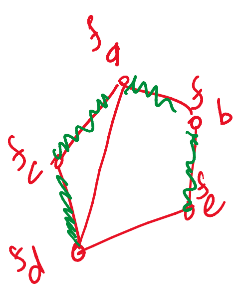
o/p: YES iff ∃ H.P. in G.

i/p: $\{f_1, \dots, f_n\}$, $C = \{(f_i, f_j) \mid f_i \ \& \ f_j$ can't be adjacent$\}$, K.

op: YES iff ∃ seq. of k flavours, at most one for each,

$f_{i_1} \to f_{i_2} \to f_{i_3} \dots \to f_{i_k}$

s.t. $2 \le l \le k$ $(f_{i_{l-1}}, f_{i_l}) \notin C$.



$F = \{f_a, f_b, f_c, f_d, f_e\}$

$F = V$

$C = \{(f_u, f_v) \mid (u, v) \notin E\}$
$= \{(f_a, f_e), (f_b, f_d), (f_b, f_c), (f_c, f_e)\}$

claim: ∃ a HP in G $\Longleftrightarrow$ ∃ stack of size $k = n$ of i.c. scoops s.t. if (f_i, f_j) adjacent then $(f_i, f_j) \notin C$.

pf: ($\Longrightarrow$)

Pf: ($\Rightarrow$) $\quad (f_i, f_j) \notin C$.

HP P in G $\longrightarrow$ $f_{v_1} \to f_{v_2} \to \cdot f_{v_i} \to f_{v_j} \to f_{v_n}$.

$v_1 \to v_2 \to v_i \to v_j \to v_n$

$f_{v_i} \to f_{v_j} \Rightarrow (v_i, v_j) \in E$

$\Rightarrow (f_{v_i}, f_{v_j}) \notin C$.

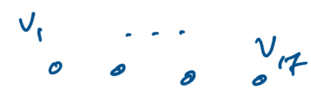
($\Leftarrow$)

$v_1 \to v_2 \dots \to v_n \longleftarrow \boxed{f_{v_1} \to f_{v_2} \to \dots \to f_{v_n}}$

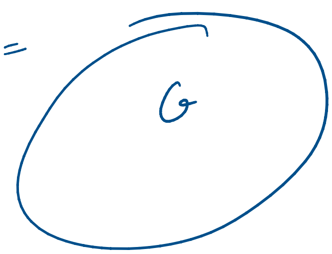
- No vertex is repeated because no flavour is repeated in the ice-cream stack.
- $v_i \to v_j \Rightarrow f_{v_i} \to f_{v_j} \Rightarrow (f_{v_i}, f_{v_j}) \notin C \Rightarrow (v_i, v_j) \in E$
  $\Rightarrow v_1 \to v_2 \to \dots \to v_n$ indeed forms a path in G.

32.A NP-hard
G_v contains path that does not visit at most 17 vertices.
undirected

HP $\le$ 17-Semi-HP $\quad v_1 \dots v_{17}$



$G' =$



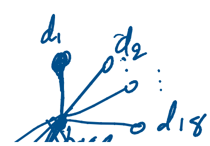
HP P in G $\longrightarrow$ P is 17-semi-HP in G'

P is a HP in G. $\longleftarrow$ P : 17-semi-HP in G'
then $v_1, \dots, v_{17}$ not on P.

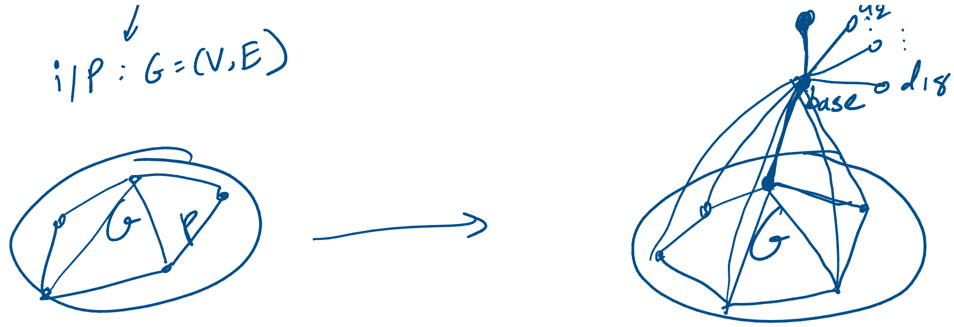
32.A' Given an undirected connected graph G, show that checking if it has a 17-Semi-HP is NP-hard.

Sol'n: Step 1: $HP \le_p$ 17-Semi-HP-Con.

i/p : $G = (V, E)$ $\qquad$ i/p : $G' = (V', E')$



i/P : $G = (V, E)$



$V' = V \cup \{base, d_1, d_2, \ldots, d_{18}\}$

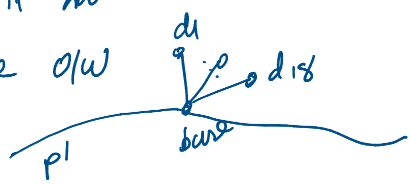
$E' = E \cup \{(u, base) \mid u \in G \text{ or } u \in \{d_1, \ldots, d_{18}\}\}$

Step 2: $\exists$ HP in G $\iff$ $\exists$ 17-semi-HP in G'

$v_1 - v_2 - v_3 - \cdots - v_n$

Pf: ($\Rightarrow$) Let P be an HP in G then construct 17-semi-HP P' in G' as follows: d_1 - base - P

($\Leftarrow$) Let P' be a 17-semi-HP in G'. Let P' be one of the d_i's. Then it has to start/end at one of the d_i's because o/w



it will have to miss all of $d_1 \ldots d_{18}$ which is 18 vertices missed!

Hence, it must look like

$d_i - base - \underbrace{v_{i_1} - v_{i_2} - \cdots - v_{i_n}}_{\in G}$

This implies $v_{i_1} - v_{i_2} - \cdots - v_{i_n}$ is an HP in G. ∎