define a subset S as follows:

for each clause $\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$

pick any j s.t. α_{ij} true

put v_{ij} in S.

Then $|S| = m = K$

& S is an indep set

check triangle edges 

check cross edges. 

Pf of [$\Leftarrow$]: Given indep set S of size $\geq K$,

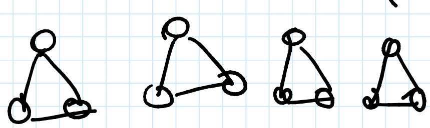
define an assignment as follows:

whenever v_{ij} is in S, set α_{ij} to true.

This assignment is consistent:

if $\alpha_{ij} = \overline{\alpha_{i'j'}}$, can't set both $\alpha_{ij}, \alpha_{i'j'}$ to true because of cross edge.

This assignment is satisfying:



m triangles

for each triangle, at most one v_{ij} is in S

since $|S| \geq K = m$, exactly one v_{ij} is in S for each i.

counting arg.

So $\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$ is true $\forall i$. $\square$

SUBSET-SUM:

Input: set $\underline{X}$ of n positive integers $\leq W$, integer W

Output: yes iff $\exists$ subset $S \subseteq X$ s.t. $\sum_{a \in S} a = W$.

e.g. $\{3, 9, 11, 20, 35, 51, 60\}$, $W = 100$

DP: $O(nW)$ time

input size:

DP: $O(nW)$ time
not polynomial

input size: $n \log W$

brute force: $O(2^n \cdot n \log W)$ time

smarter: $\sim O(2^{n/2} \log W)$

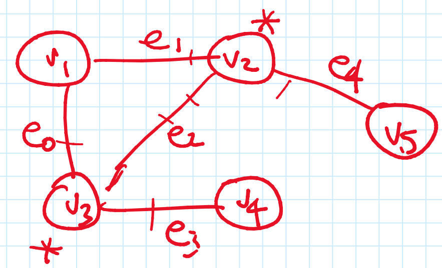
Thm Subset-Sum is NP-complete.

Pf: ① Subset-Sum $\in$ NP. ✓

② Reduce from Vertex-Cov to Subset-Sum:

Given graph $G = (V, E)$, number K,
construct set X of integers and value W as follows:

example: given G:



$K = 2$

first idea: incidence matrix

$$
\begin{array}{c|cccccc|l}
 & & e_4 & e_3 & e_2 & e_1 & e_0 & \\
\hline
v_1 & 1 & 0 & 0 & 0 & 1 & 1 & = a_1 \\
* v_2 & 1 & 1 & 0 & 1 & 1 & 0 & = a_2 \\
* v_3 & 1 & 0 & 1 & 1 & 0 & 1 & = a_3 \\
v_4 & 1 & 0 & 1 & 0 & 0 & 0 & = a_4 \\
v_5 & 1 & 1 & 0 & 0 & 0 & 0 & = a_5 \\
 & & & & & & 1 & = b_0 \\
 & & & & & 1 & 0 & = b_1 \\
 & & & & 1 & 0 & 0 & = b_2 \\
 & & & 1 & 0 & 0 & 0 & = b_3 \\
 & & 1 & 0 & 0 & 0 & 0 & = b_4 \\
\hline
 & K & 2 & 2 & 2 & 2 & 2 & = W = a_2 + a_3 + b_0 + b_1 + b_3 + b_4
\end{array}
$$

$K \; 2 \; 2 \; 2 \; 2 \; 2 = W = a_2 + a_3 + b_0 + b_1 + b_3 + b_4$

Suppose $V = \{v_1, \dots, v_n\}$ let $c_{ij} = \begin{cases} 1 & \text{if } e_j \text{ incid to } v_i \\ 0 & \text{else} \end{cases}$

$E = \{e_0, \dots, e_{m-1}\}$

define $X = \{a_1, \dots, a_n, b_0, \dots, b_{m-1}\}$

where $a_i = 10^m + \sum_{j=0}^{m-1} c_{ij} 10^j$

$b_j = 10^j$

$$W = K \cdot 10^m + \sum_{j=0}^{m-1} 2 \cdot 10^j$$

each number has $O(m)$ digits/bits

This construction from (G, K) to (X, W) takes polytime.

<u>Correctness</u>: $\exists$ vertex cover S of G of size K

$\iff \exists$ subset $S' \subseteq X$ summing to W.

Pf of $(\Rightarrow)$: Suppose S is a VC of size K.

Define $S' = \{a_i : v_i \in S\} \cup \{b_j : e_j \text{ is incid to exactly one vertex of } S\}$

then sum of S' is W.

Pf of $(\Leftarrow)$:

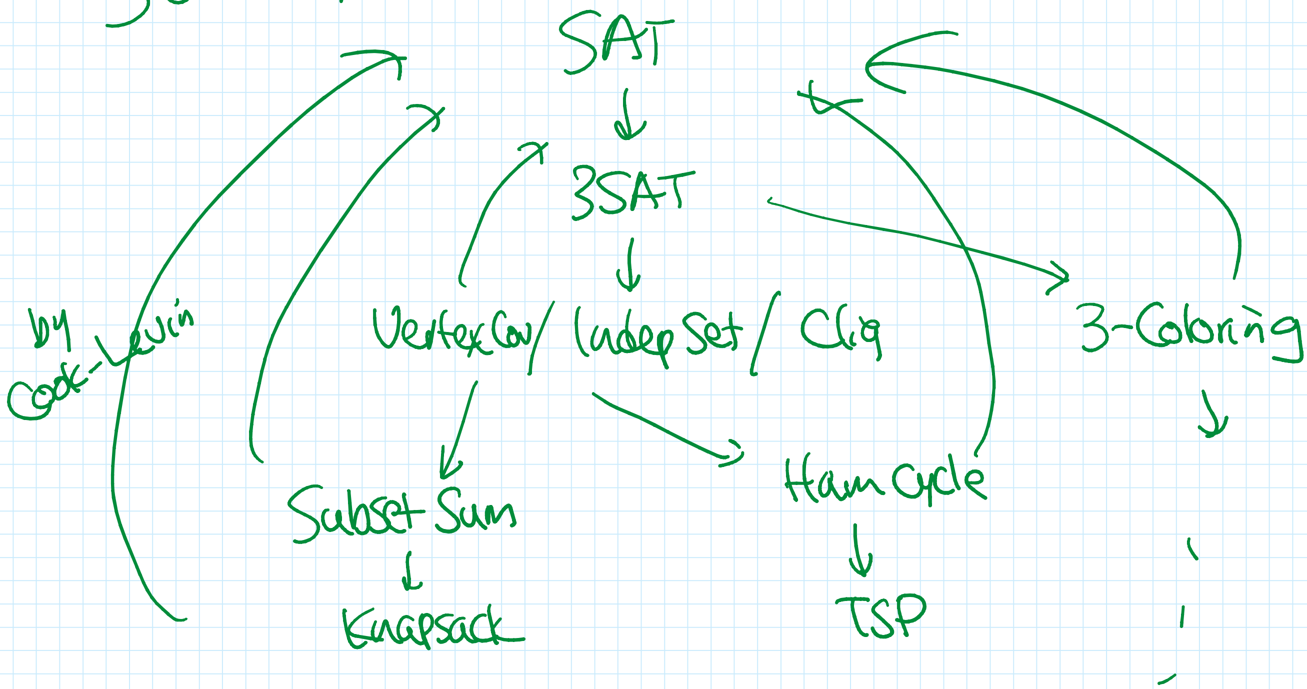
Suppose $S' \subseteq X$ sums to W.

Define $S = \{v_i : a_i \in S'\}$.

Then $|S| = K$ because of m^{th} digit

S is a vertex cover... $\square$

Summary:


SAT
3SAT
by Cook-Levin
VertexCov / IndepSet / Cliq
3-Coloring
Subset Sum
Ham Cycle
Knapsack
TSP