Thursday, March 7, 2024 10:51 AM

More DP

"Try all possibilities"

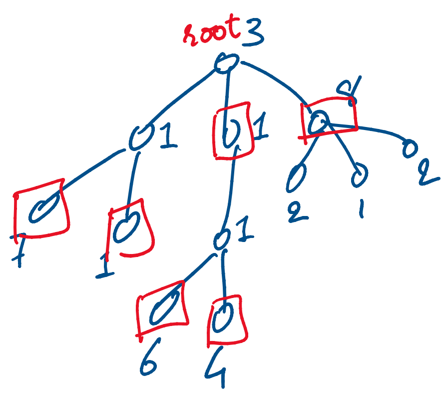
⇓

Recursive Alg'm → Memoize → Evaluation order

↓

Iterative alg'm

Probl: Max Weight Independent Set (MWIS) on trees.



Given: $T = (V, E)$ acyclic

$v \in V,\ w(v) \geq 0$

Find $S \subseteq V$ that maximizes $\sum_{v \in S} w(v)$

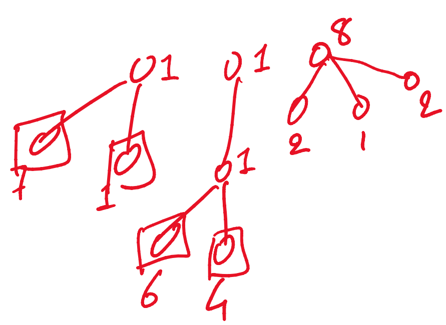
s.t. $\forall u, v \in S,\ (u, v) \notin E$.

$8 + 1 + 1 + 7 + 6 + 4 = 27$

★ Observations: v = root

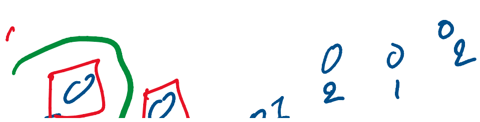
- either v is in opt. sol or not.
- v not in opt sol'n

  recurse on subtrees rooted at children of v.

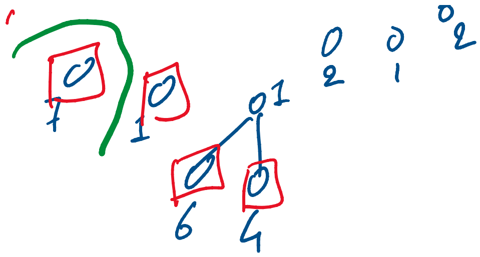


- v is in opt sol'n

  recurse on subtrees rooted at grand children of v



rooted at grand children of v



① **Define Subproblems:** $v \in V$

$A(v)$ = weight of MWIS in the subtree rooted at v.

Ans: $A(\text{root})$.

② **Base case:** if v is a leaf $A(v) = w(v)$

otherwise.

Recursive formula:

$$A(v) = \max \left\{ \sum_{\substack{u \text{ child} \\ \text{of } v}} A(u) \, , \quad w(v) + \sum_{\substack{u \text{ grandchild} \\ \text{of } v}} A(u) \right\}$$

③ **Evaluation Order:**

in Post order traversal of vertices.

Psuedocode:

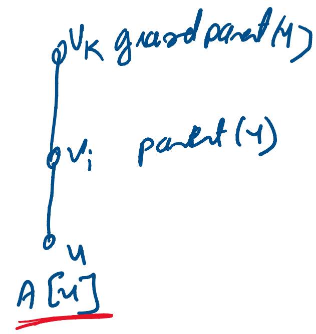
$[v_1, \ldots v_i, v_n]$ = post-order traversal T.

$O(n)\{$ 1. $[v_1, \dots, v_i, v_n]$ = post-order
$\quad 10 \quad 100 \quad 1 \quad 30$

2. for $i = 1$ to n
3. if v_i is a leaf then $A[v_i] = w(v_i)$
4. Else

$$O(n)\left\{ \underline{A[v_i]} = \max \left\{ \underline{\sum_{\substack{u \text{ child} \\ \text{of } v_i}} A[u]}, \; w(v_i) + \underline{\sum_{\substack{u \text{ grandchild} \\ \text{of } v_i}} A[u]} \right\} \right.$$

5. Return $A[v_n]$



Run Time: # subproblems $= O(n)$

time per subproblem $= O(n)$

$\Rightarrow \boxed{O(n^2)}$ Time.

Better Analysis: How many time $A[u]$ appears on RHS. for a given $u \in V$

Twice! Once for the parent of u & once for the grand parent of u.

$\Rightarrow O(n)$

$\hookrightarrow$ **Amortized analysis.**

Prob 2: Subset Sum

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Given numbers $a_1, a_2, \ldots, a_n$ & T

check if $\exists S \subseteq \{1, \ldots, n\}$ s.t. $\sum_{i \in S} a_i = T$.

O/P : TRUE or FALSE.

e.g. 1, 10, 5, 15, 20, 2, 30, $\boxed{13}$ $\quad$ $\underline{T = 22}$

TRUE

① **Define Subproblem:** $0 \le i \le n$, $0 \le t \le T$

$ss(i,t) =$ true iff $a_1, \ldots a_i$ has a subset summing to t.

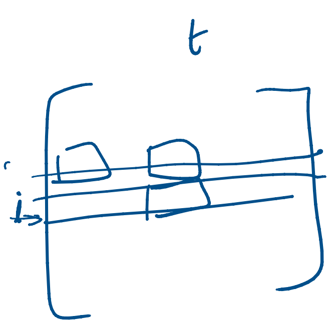
Base case:

$ss(0,0) =$ true

$ss(0,t) =$ false $\quad \forall t \ge 1$.

Recursive formula:

$\underbrace{a_1, \ldots, \boxed{a_i}}$ $\qquad\qquad$ t



$$ss(i,t) = \begin{cases} ss(i-1,t) \lor ss(i-1, t-a_i) & \text{if } a_i \le t \\ ss(i-1,t) \end{cases}$$

Evaluation order:

increasing order of i.

for each i, any order of t.

Run Time: # subproblem : $O(nT)$

time per subproblem: $O(1)$

$\Rightarrow$ $O(nT)$ time.

i/p size : $O(n \cdot \lg T)$

$T = 2^{\lg T}$

exp. time.

Rmk: T can be very large unfortunately.

poly in n & $\log T$? Big open que

$P = NP$?

Variants:

Prob 3: Parsing:

Given a CFG $G = (V, T, P, S)$ &

string $x = a_1 \dots a_n \in T^*$

Check if $x \in L(G)$?

e.g.

CNF
$$\begin{cases} S \to AS \mid AB \\ A \to BC \\ B \to SA \mid AC \mid 1 \\ C \to AB \mid CC \mid 0 \end{cases}$$

$x = 1011011 \in L(G)$?

Assume: every production rule in P is of type
$$\left.\begin{matrix} A \to BC \\ A \to a \end{matrix}\right\} \text{CNF}$$

★ CKY Alg'm (Cocke - Younger - Kassami '70)

① Define subprob.: $1 \le i \le j \le n$, $A \in V$

$f(i, j, A)$ = true iff $a_i \dots a_j$ can be generated starting at A.

Ans: $f(1, n, S)$

Base case: a_i be generated by A?

Base case: a_i be generated by A:

$$f(i,i,A) = \text{true iff } (A \to a_i) \in P.$$
$$= \text{false o.w.}$$

Recursive formula: $i < j$

$f(i, j, A)$: $\quad \underbrace{a_i \ldots}_{B} \mid \underbrace{a_j}_{C}$ be gen. by starting at A!

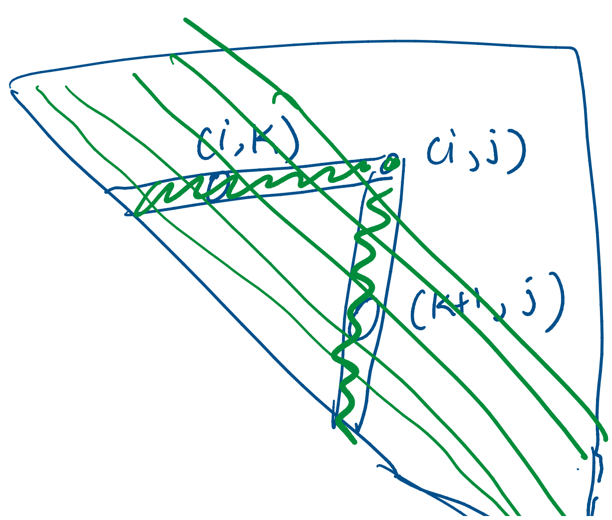
Choices:
- which production rule to use $A \to BC$
- split $\underbrace{a_i \ldots a_k}_{B} \underbrace{a_{k+1} \ldots a_j}_{C}$, $i \le k \le j-1$

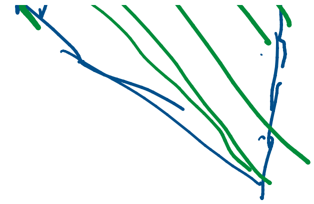
$$f(i,j,A) = \underbrace{\bigvee_{A \to BC \text{ in } P}}_{O(|P|)} \underbrace{\bigvee_{k \in \{i,\ldots,j-1\}}}_{O(n)} \left(f(i,k,B) \wedge f(k+1,j,C) \right)$$

j fixed A

③ Evaluation order:

increasing order of $(j-i)$





Run time:

\# subproblems: $O(n^2 \cdot |V|)$

time per subprob: $O(n \cdot |P|)$

$\Rightarrow \quad O(n^3 \cdot |V| \cdot |P|)$

Rmk: Valiant '75 $\quad O(|P|\, n^{2.373})$

for most PL there are linear-time parsing alg'm.