February 8, 2024 10:39 AM

Regular $\longleftrightarrow$ NFA $\longleftrightarrow$ DFA

but lots of langs not regular

e.g. $\{0^n 1^n : n \geq 0\}$

Context-Free Languages (CFL)

intuitively: langs generated by recursive replacement rules

Ex 1 $\{0^n 1^n : n \geq 0\}$

Rules:
$$S \to 0S1$$
$$S \to \varepsilon$$

e.g. to generate 000111,

$S \rightsquigarrow 0S1 \rightsquigarrow 00S11$
$\rightsquigarrow 000S111$
$\rightsquigarrow 000111$

Ex 2 all even-length palindromes in $\{0,1\}^*$
$= \{ww^R : w \in \{0,1\}^*\}$

Rules:
$$S \to 0S0$$
$$S \to 1S1$$
$$S \to \varepsilon$$

to include odd-length

$S \to 0 \quad S \to 1$

e.g. 011110: $S \rightsquigarrow 0S0 \rightsquigarrow 01S10$
$\rightsquigarrow 011S110$

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$\rightsquigarrow 011S110$
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Formal Def'n A context-free grammar (CFG)

is $G = (V, \Sigma, P, S)$

where V is finite set of variables ("non-terminals")

Σ is finite alphabet ("terminals")

P is finite set of rules ("productions")

of the form $A \to \alpha$

where $A \in V$, $\alpha \in (\Sigma \cup V)^*$

$S \in V$ is the start symbol.

Ex: $V = \{S\}$, $\Sigma = \{0,1\}$

$P = \{ S \to 0S1, \ S \to \varepsilon \}$

Def $\alpha_1 \underset{G}{\rightsquigarrow} \alpha_2$ (α_1 derives α_2 in one step)

iff $\alpha_1 = \beta A \delta$ and

$\alpha_2 = \beta \alpha \delta$ and

$A \to \alpha$ is in P

for some $A \in V$, $\beta, \delta \in (\Sigma \cup V)^*$.

Ex $\underbrace{00}_{\beta}\underbrace{S}_{A}\underbrace{11}_{\delta} \underset{G}{\rightsquigarrow} \underbrace{00}_{\beta}\underbrace{0S1}_{\alpha}\underbrace{11}_{\delta}$.

$\underbrace{0}_{\beta}S\underbrace{0S11}_{\delta} \underset{G}{\rightsquigarrow} \underbrace{0}_{\beta}0S1\underbrace{0S11}_{\delta}$

$\underbrace{0S0}_{\beta}\underbrace{S}_{A}\underbrace{11}_{\delta} \underset{G}{\rightsquigarrow} \underbrace{0S0}_{\beta}\underbrace{0S1}_{\alpha}\underbrace{11}_{\delta}$

Def $\alpha_1 \overset{k}{\underset{G}{\rightsquigarrow}} \alpha_2$ (α_1 derives α_2 in k steps)

iff ($\alpha_1 = \alpha_2$ if $k = 0$

iff $\begin{cases} \alpha_1 = \alpha_2 & \text{if } k = 0 \\ \alpha_1 \underset{G}{\rightsquigarrow} \beta \text{ and } \beta \overset{k-1}{\underset{G}{\rightsquigarrow}} \alpha_2 & \text{if } k > 0 \end{cases}$

for some $\beta \in (\Sigma \cup V)^*$.

$\alpha_1 \overset{*}{\underset{G}{\rightsquigarrow}} \alpha_2$ iff $\alpha_1 \overset{k}{\underset{G}{\rightsquigarrow}} \alpha_2$ for some $k \geq 0$.

Def $L(G) = \{ x \in \Sigma^* : S \overset{*}{\underset{G}{\rightsquigarrow}} x \}$

lang generated by G

L is a CFL iff $L = L(G)$ for some CFG G.

Ex a) $\underbrace{\overbrace{(1+01)^*}^{C}\,\overbrace{(1+10)^*}^{D}}_{A} + \underbrace{1^*0}_{B}$

$S \to CD \mid B$

shorthand $S \to A \mid B$ for $S \to A$, $S \to B$

$C \to 1C \mid 01C \mid \varepsilon$

$D \to 1D \mid 10D \mid \varepsilon$

$B \to 1B \mid 0$

e.g. $0110 \in L(G)$

$S \rightsquigarrow CD \rightsquigarrow 01CD$

$\rightsquigarrow 01D$

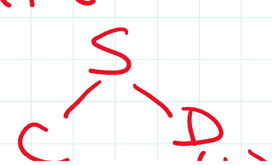
$\rightsquigarrow 0110D$

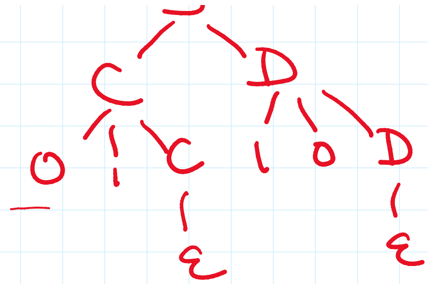
$\rightsquigarrow 0110.$

$\cancel{CO \to CO}$

parse tree

derivation tree





(prog langs appl: <stmt> → while <expr> do <stmts> end
<stmts> → <stmt> ; <stmts>
⋮)

b) $\{0^i 1^j 2^k : j > i + k\}$

idea. $\underbrace{0^i 1^i}_{A} \underbrace{1^{j-i-k}}_{B} \underbrace{1^k 2^k}_{C}$

$$S \to ABC$$
$$A \to 0A1 \mid \varepsilon$$
$$C \to 1C2 \mid \varepsilon$$
$$B \to 1B \mid 1$$

c) $\{0^i 1^i 2^i : i \geq 0\}$

not possible (how to prove?)

d) $\{ww : w \in \{0,1\}^*\}$.

not possible

e) $\{x \in \{0,1\}^* : x \text{ is not a palindrome}\}$

01$\boxed{0}$00$\boxed{1}$10

$$S \to 0S0 \mid 1S1 \mid 0A1 \mid 1A0$$
$$A \to 0A \mid 1A \mid \varepsilon.$$

f) Complement of $\{ww : w \in \{0,1\}^*\}$

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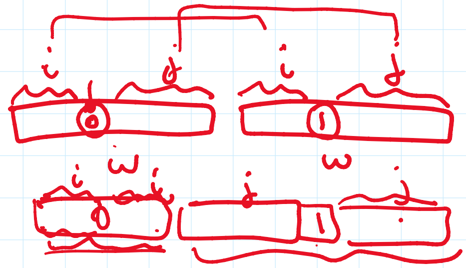
possible!



$$
\begin{cases}
S \rightarrow AB \mid BA \mid C \\
A \rightarrow DAD \mid 0 \\
B \rightarrow DBD \mid 1 \\
C \rightarrow DDC \mid D \\
D \rightarrow 0 \mid 1
\end{cases}
$$

← odd-length string