Kleene's Thm

regular $\rightleftarrows$ NFA $\rightleftarrows$ DFA

Are there languages that are not regular?
Yes.

How to prove a language is not regular??

"Fooling Set" Method

Motivating Ex 0

$L = \{0^n 1^n : n \geq 0\}$ is not regular.

Pf: By contradiction.
Suppose L is regular.
Then L is accepted by some DFA $M = (Q, \Sigma, s, \delta, A)$

vague idea - machine needs to "remember" n

Claim $\delta^*(s, 0^i) \neq \delta^*(s, 0^j)$ $\forall i, j$ with $i \neq j$.

Pf: if $\delta^*(s, 0^i) = \delta^*(s, 0^j)$,

$$\delta^*(\underbrace{\delta^*(s,0^i)}, 1^i) = \delta^*(\underbrace{\delta^*(s,0^j)}, 1^i)$$

$$\underbrace{\delta^*(s, 0^i 1^i)}_{\in A} = \underbrace{\delta^*(s, 0^j 1^i)}_{\notin A}$$

$\boxed{\begin{array}{l}\delta^*(s, xy) \\ = \delta^*(\delta^*(s,x), y). \\ \text{by induction.}\end{array}}$

Contradiction! $\square$

By Claim, $\delta^*(s, 0^0)$, $\delta^*(s, 0^1)$, $\delta^*(s, 0^2)$, ... are all distinct

$\Rightarrow$ Q infinite: Contradiction! $\square$

Generalize ...

Def Given language L,

a string $x, y \in \Sigma^*$ are <u>distinguishable</u> iff

$\boxed{\exists} w \in \Sigma^*$, $(xw \in L \;\&\; yw \notin L)$

or $(xw \notin L \;\&\; yw \in L)$.

Def F is a <u>fooling set</u> if

$\boxed{\forall} x, y \in F$ with $x \neq y$,

x and y are distinguishable.

Thm If L has an infinite fooling set, then L is not regular. (iff)

Pf: as in Ex 0.

$(\delta^*(s,x) \neq \delta^*(s,y) \quad \forall x, y \in F$ with $x \neq y)$

$\ldots$ $\square$

Ex a) $L = \{x \in \{0,1\}^* : \#_0(x) = \#_1(x)\}$ is not regular.

Pf: Let $F = \{0^i : i \geq 0\}$.

Given 2 arbitrary strings $x, y \in F$ $(x \neq y)$,

$x = 0^i$, $y = 0^j$ $(i \neq j)$.

Pick $w = 1^i$ (Pick $w = 1^{i+1}0$)

$\Rightarrow$ $xw = 0^i 1^i \in L$

$yw = 0^j 1^i \notin L$

$\Rightarrow$ x, y distinguishable

~~$w = 1$~~

~~$xw = 0^i 1 \notin L$~~

~~$yw = 0^j 1 \notin L$~~

you choose

they choose

you choose

Thus, F is a fooling set, infinite. $\square$

b) $L = \{\text{all palindromes}\}$

$= \{w \in \{0,1\}^* : w = w^R\}$

0110

01011010

101

is not regular.

Pf: Let $F = \{0^i : i \geq 0\}$.

Given 2 arbitrary strings $x, y \in F$ $(x \neq y)$

$x = 0^i$, $y = 0^j$ $(i \neq j)$

Pick $w = 10^i$

$\Rightarrow$ $xw = 0^i 1 0^i \in L$ since $i \neq j$

$yw = 0^j 1 0^i \notin L$

$\Rightarrow$ x, y distinguishable

$\therefore$ F is fooling set & infinite. $\square$

c) $L = \{0^{n^2} : n \geq 0\}$ is not regular.

Pf: Let $F = \{0^i : i \geq 0\}$.

Given 2 arbitrary strings $x, y \in F$ with $x \neq y$

$x = 0^i$, $y = 0^j$, with $i \neq j$.

w.l.o.g. say $i < j$.

Pick $w = 0^{j^2 - i}$

$\Rightarrow$ $xw = 0^{j^2} \in L$

$yw = 0^{j^2 - i + j} \notin L$

why?

$j^2 < j^2 - i + j < (j+1)^2$

because

$j^2 - i + j \leq j^2 + j < j^2 + 2j + 1$

Thus, F is fooling set, infinite. $\square$

d) $L = \{ 0^p : p \text{ prime} \}$ is not regular.

Pf: Let $F = \{ 0^i : i \geq 0 \}$.

Given $x, y \in F$ with $x \neq y$,

$x = 0^i$, $y = 0^j$ $i \neq j$

w.l.o.g. say $i < j$.

Pick some prime $p > i$.

(first try $w = 0^{p-i}$ $xw \in L$ ✓
$yw = 0^{p+j-i}$ not work if $p+j-i$ is prime

try $w = 0^{p+j-2i}$ $xw = 0^{p+j-i} \in L$ ✓
$yw = 0^{p+2j-2i}$ not work if $p+2j-2i$ is prime

⋮)

Let k be smallest s.t. $(1 \leq k \leq p)$
$p + k(j-i)$ is composite.

Pick $w = 0^{p+k(j-i)-j}$

$\Rightarrow$ $xw = 0^{p+k(j-i)-j+i}$

$= 0^{\underbrace{p+(k-1)(j-i)}_{\text{prime}}} \in L$

$yw = 0^{\underbrace{p+k(j-i)}_{\text{composite}}} \notin L$.

Then F is a fooling set, infinite. $\square$

Modified Thm If L has a finite fooling set F of size n,

then any DFA for L needs $\geq n$ states.

Ex $L = \{$ all strings in $\{0,1\}^*$ whose 5^{th} last symbol

Ex

$L = \{$ all strings in $\{0,1\}^*$ whose 5^{th} last symbol is $0\}$.

32 states

let $F = \{$ all strings with 5 chars$\}$ $|F| = 32$

$x = a_1 a_2 a_3 a_4 a_5 \overbrace{00}^{w}$

$y = b_1 b_2 b_3 b_4 b_5\ 00$

$a_i \neq b_i$ for some i.

$w = 0^{i-1}$