Prove that each of the following problems is NP-hard.

1 Prove that the following problem is NP-hard: Given an undirected graph G, find *any* integer $k > 374$ such that G has a proper coloring with k colors but G does not have a proper coloring with $k - 374$ colors.

Solution:

Let G' be the union of 374 copies of G, with additional edges between *every* vertex of each copy and *every* vertex in *every* other copy. Given G, we can easily build G' in polynomial time by brute force. Let $\chi(G)$ and $\chi(G')$ denote the minimum number of colors in any proper coloring of G, and define $\chi(G')$ similarly.

- $\Longrightarrow$ Fix any coloring of G with $\chi(G)$ colors. We can obtain a proper coloring of G' with $374 \cdot \chi(G)$ colors, by using a distinct set of $\chi(G)$ colors in each copy of G. Thus, $\chi(G') \leq 374 \cdot \chi(G)$.
- $\Longleftarrow$ Now fix any coloring of G' with $\chi(G')$ colors. Each copy of G in G' must use its own distinct set of colors, so at least one copy of G uses at most $\lfloor \chi(G')/374 \rfloor$ colors. Thus, $\chi(G) \leq \lfloor \chi(G')/374 \rfloor$.

These two observations immediately imply that $\chi(G') = 374 \cdot \chi(G)$. It follows that if k is an integer such that $k - 374 < \chi(G') \leq k$, then $\chi(G) = \chi(G')/374 = \lceil k/374 \rceil$. Thus, if we could compute such an integer k in polynomial time, we could compute $\chi(G)$ in polynomial time. But computing $\chi(G)$ is NP-hard!

2 A ***bicoloring*** of an undirected graph assigns each vertex a set of *two* colors. There are two types of bicoloring: In a *weak* bicoloring, the endpoints of each edge must use *different* sets of colors; however, these two sets may share one color. In a *strong* bicoloring, the endpoints of each edge must use *distinct* sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

2.A. Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.

Solution:

It suffices to prove that deciding whether a graph has a weak bicoloring with three colors is NP-hard, using the following trivial reduction from the standard 3COLOR problem.

Let G be an arbitrary undirected graph. I claim that G has a proper 3-coloring if and only if G has a weak bicoloring with 3 colors.

- $\Rightarrow$ Suppose G has a proper coloring using the colors red, green, and blue. We can obtain a weak bicoloring of G using only the colors cyan, magenta, and yellow by recoloring each red vertex with $\{\text{magenta}, \text{yellow}\}$, recoloring each blue vertex with $\{\text{magenta}, \text{cyan}\}$, and recoloring each green vertex with $\{\text{yellow}, \text{cyan}\}$.
- $\Leftarrow$ Suppose G has a weak bicoloring using the colors cyan, magenta, yellow. Then we can obtain a proper 3-coloring of G by defining red $= \{\text{magenta}, \text{yellow}\}$, defining blue $= \{\text{magenta}, \text{cyan}\}$, and defining green $= \{\text{yellow}, \text{cyan}\}$.

More generally, for any integer k and any graph G, every weak k-bicoloring of G is also a proper $\binom{k}{2}$-coloring of G, and vice versa.

2.B. Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.

Solution:

It suffices to prove that deciding whether a graph has a strong bicoloring with five colors is NP-hard, using the following reduction from the standard 3COLOR problem.

Let $G = (V, E)$ be an arbitrary undirected graph. We build a new graph $G' = (V', E')$ as follows:

- Initialize $V' = V$. Add a new vertex s to V'.
- Initialize $E' = \emptyset$. For each $v \in V$, add edge sv to E'.
- For each $uv \in E$, add two new vertices x_{uv} and y_{uv} to V', and add three edges ux_{uv}, $x_{uv}y_{uv}$, and $y_{uv}v$ to E'.

I claim that G has a proper 3-coloring if and only if G' has a strong bicoloring with five colors.

$\Rightarrow$ Suppose G has a proper 3-coloring with colors red, green, and blue. Then we define a strong bicoloring of G' with colors $1, 2, 3, 4, 5$ as follows:

- Let $\mathit{color}(s) = \{4, 5\}$.
- For each red $v \in V$, let $\mathit{color}(v) = \{1, 2\}$.
- For each green $v \in V$, let $\mathit{color}(v) = \{2, 3\}$.
- For each blue $v \in V$, let $\mathit{color}(v) = \{1, 3\}$.
- For every $uv \in E$, if u is red and v is green, let $\mathit{color}(x_{uv}) = \{3, 4\}$ and $\mathit{color}(y_{uv}) = \{1, 5\}$.
- For every $uv \in E$, if u is red and v is blue, let $\mathit{color}(x_{uv}) = \{3, 4\}$ and $\mathit{color}(y_{uv}) = \{2, 5\}$.
- For every $uv \in E$, if u is green and v is blue, let $\mathit{color}(x_{uv}) = \{1, 4\}$ and $\mathit{color}(y_{uv}) = \{2, 5\}$.

It is easy to check that every pair of adjacent vertices of G' has disjoint color sets.

$\Leftarrow$ Suppose G' has a strong bicoloring with five colors. Without loss of generality (by renumbering), suppose $\mathit{color}(s) = \{4, 5\}$. We define a 3-coloring in G as follows: for each $v \in V$,

- If $\mathit{color}(v) = \{1, 2\}$, then color v red.
- If $\mathit{color}(v) = \{2, 3\}$, then color v green.
- If $\mathit{color}(v) = \{1, 3\}$, then color v blue.

These are the only possibilities, since $\mathit{color}(v)$ is disjoint from $\mathit{color}(s) = \{4, 5\}$.

We now check that this 3-coloring is proper. Consider an edge $uv \in E$. For the sake of contradiction, suppose u and v have the same color in G. Then $\mathit{color}(u) = \mathit{color}(v)$ in G'. But since $ux_{uv}, y_{uv}v \in E'$, we have $\mathit{color}(x_{uv})$ and $\mathit{color}(y_{uv})$ contained in a set $\{1, 2, 3, 4, 5\} - \mathit{color}(u)$ with 3 elements. But since $x_{uv}y_{uv} \in E'$, $\mathit{color}(x_{uv})$ and $\mathit{color}(y_{uv})$ are disjoint and together have 4 elements: a contradiction.