Prove that each of the following problems is NP-hard.

1 Given an undirected graph G, does G contain a simple path that visits all but 374 vertices?

Solution:

We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph G, let H be the graph obtained from G by adding 374 isolated vertices. Call a path in H ***almost-Hamiltonian*** if it visits all but 374 vertices. I claim that G contains a Hamiltonian path if and only if H contains an almost-Hamiltonian path.

- $\Rightarrow$ Suppose G has a Hamiltonian path P. Then P is an almost-Hamiltonian path in H, because it misses only the 374 isolated vertices.
- $\Leftarrow$ Suppose H has an almost-Hamiltonian path P. This path must miss all 374 isolated vertices in H, and therefore must visit every vertex in G. Every edge in H, and therefore every edge in P, is also an edge in G. We conclude that P is a Hamiltonian path in G.

Given G, we can easily build H in polynomial time by brute force.

2 Given an undirected graph G, does G have a spanning tree with at most 374 leaves?

Solution:

We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem.[1] Given an arbitrary graph G, let H be the graph obtained from G by adding the following vertices and edges:

- First we add a vertex z with edges to every other vertex in G.
- Then we add 373 vertices $\ell_1, \ldots, \ell_{373}$, each with edges to z and nothing else.

Call a spanning tree of H ***almost-Hamiltonian*** if it has at most 374 leaves. I claim that G contains a Hamiltonian path if and only if H contains an almost-Hamiltonian spanning tree.

- $\Rightarrow$ Suppose G has a Hamiltonian path P. Suppose P starts at vertex s and ends at vertex t. Let T be subgraph of H obtained by adding the edge tz and all possible edges $z\ell_i$. Then T is a spanning tree of H with exactly 374 leaves, namely s and all 373 new vertices ℓ_i.
- $\Leftarrow$ Suppose H has an almost-Hamiltonian spanning tree T. Every node ℓ_i is a leaf of T, so T must consist of the 373 edges $z\ell_i$ and a simple path from z to some vertex s of G. Let t be the only neighbor of z in T that is not a leaf ℓ_i, and let P be the unique path in T from s to t. This path visits every vertex of G; in other words, P is a Hamiltonian path in G.

Given G, we can easily build H in polynomial time by brute force.

3 Recall that a 5-coloring of a graph G is a function that assigns each vertex of G a "color" from the set $\{0, 1, 2, 3, 4\}$, such that for any edge uv, vertices u and v are assigned different "colors". A 5-coloring is ***careful*** if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5). Prove that deciding whether a given graph has a careful 5-coloring is NP-hard.

Solution:

We prove that careful 5-coloring is NP-hard by reduction from the standard 5COLOR problem.

Given a graph G, we construct a new graph H by replacing each edge in G with a path of length three. I claim that H has a careful 5-coloring if and only if G has a (not necessarily careful) 5-coloring.

$\Longleftarrow$ Suppose G has a 5-coloring. Consider a single edge uv in G, and suppose $\mathit{color}(u) = a$ and $\mathit{color}(v) = b$. We color the path from u to v in H as follows:

- If $b = (a+1) \bmod 5$, use colors $(a,\ (a+2) \bmod 5,\ (a-1) \pmod 5,\ b)$.
- If $b = (a-1) \bmod 5$, use colors $(a,\ (a-2) \bmod 5,\ (a+1) \pmod 5,\ b)$.
- Otherwise, use colors $(a,\ b,\ a,\ b)$.

In particular, every vertex in G retains its color in H. The resulting 5-coloring of H is careful.

$\Longrightarrow$ On the other hand, suppose H has a careful 5-coloring. Consider a path (u, x, y, v) in H corresponding to an arbitrary edge uv in G. Without loss of generality, say $\mathit{color}(u) = 0$; there are exactly eight careful colorings of this path with $\mathit{color}(u) = 0$, namely: $(0, 2, 0, 2)$, $(0, 2, 0, 3)$, $(0, 2, 4, 1)$, $(0, 2, 4, 2)$, $(0, 3, 0, 3)$, $(0, 3, 0, 2)$, $(0, 3, 1, 3)$, $(0, 3, 1, 4)$. It follows immediately that $\mathit{color}(u) \neq \mathit{color}(v)$. Thus, if we color each vertex of G with its color in H, we obtain a valid 5-coloring of G.

Given G, we can clearly construct H in polynomial time.