Prove that each of the following languages is ***not*** regular.

1. $\left\{0^{2^n} \mid n \geq 0\right\}$

Solution (fooling set $F = L$): Let $F = L = \left\{0^{2^n} \mid n \geq 0\right\}$.

Let x and y be arbitrary distinct elements of F.

Then $x = 0^{2^i}$ and $y = 0^{2^j}$ for some non-negative integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let $z = 0^{2^i}$.

- $xz = 0^{2^i}0^{2^i} = 0^{2^{i+1}} \in L$.
- $yz = 0^{2^j}0^{2^i} = 0^{2^i+2^j}$. The integer $2^i + 2^j$ lies strictly between 2^j and 2^{j+1} (because $i < j$) and thus is not a power of 2. It follows that $yz \notin L$.

Because $xz \in L$ and $yz \notin L$, the suffix z distinguishes x and y.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (fooling set $F = 0^*$): Let $F = 0^* = \{0^n \mid n \geq 0\}$.

Let x and y be arbitrary distinct elements of F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let r be any integer such that $2^r > j$, and let $z = 0^{2^r-i}$.

Then $xz = 0^i0^{2^r-i} = 0^{2^r} \in L$.

But $yz = 0^j0^{2^r-i} = 0^{2^r+j-i} \notin L$, because $2^r + j - i$ is not a power of 2:

$$\begin{aligned}
2^r < \mathbf{2^r + j - i} & & [i < j]\\
\leq\ 2^r + j & & [i \geq 0]\\
<\ 2^r + 2^r & & [j < 2^r]\\
=\ 2^{r+1} & & [\text{math}]
\end{aligned}$$

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (fooling set $F = 0^*$, but distinguish the other way):

Let $F = 0^* = \{0^n \mid n \geq 0\}$.

Let x and y be arbitrary distinct elements of F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let r be any integer such that $2^{r-1} > j$, and let $z = 0^{2^r-j}$.

Then $xz = \mathtt{0}^i\mathtt{0}^{2^r-j} = \mathtt{0}^{2^r-j+i} \notin L$, because $2^r - j + i$ is not a power of 2:

$$\begin{aligned}
2^{r-1} &= 2^r - 2^{r-1} && [\text{math}]\\
&< 2^r - j && [2^{r-1} > j]\\
&\le \boldsymbol{2^r - j + i} && [i \ge 0]\\
&< 2^r && [i < j]
\end{aligned}$$

But $yz = \mathtt{0}^j\mathtt{0}^{2^r-j} = \mathtt{0}^{2^r} \in L$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

2. $\{0^{2n}1^n \mid n \geq 0\}$

Solution (fooling set $F = (00)^*$): Let F be the language $(00)^*$.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^{2i}$ and $y = 0^{2j}$ for some non-negative integers $i \neq j$.

Let $z = 1^i$.

Then $xz = 0^{2i}1^i \in L$.

And $yz = 0^{2j}1^i \notin L$, because $i \neq j$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ∎

Solution (fooling set $F = 0^*$): Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i1^i$.

Then $xz = 0^{2i}1^i \in L$.

And $yz = 0^{i+j}1^i \notin L$, because $i + j \neq 2i$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ∎

3. $\{0^m 1^n \mid m \neq 2n\}$

Solution (fooling set $F = (00)^*$): Let F be the language $(00)^*$.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^{2i}$ and $y = 0^{2j}$ for some non-negative integers $i \neq j$.

Let $z = 1^i$.

Then $xz = 0^{2i}1^i \notin L$.

And $yz = 0^{2j}1^i \in L$, because $i \neq j$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (fooling set $F = 0^*$): Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i 1^i$.

Then $xz = 0^{2i}1^i \notin L$.

And $yz = 0^{i+j}1^i \in L$, because $i + j \neq 2i$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (reduction via closure): If L were regular, then the language

$$0^*1^* \setminus L \;=\; \{0^m 1^n \mid m = 2n\} \;=\; \left\{0^{2n}1^n \;\middle|\; n \geq 0\right\}$$

would also be regular, because regular languages are closed under complement. But we just proved that $\left\{0^{2n}1^n \;\middle|\; n \geq 0\right\}$ is not regular in problem 2. ■

4. Strings over $\{0, 1\}$ where the number of 0s is exactly twice the number of 1s.

Solution (fooling set $F = 1^*$): Let F be the language 1^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 1^i$ and $y = 1^j$ for some non-negative integers $i \neq j$.

Let $z = 0^{2i}$.

Then $xz = 1^i0^{2i} \in L$.

And $yz = 1^j0^{2i} \notin L$, because $i \neq j$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (fooling set $F = 0^*$): Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i1^i$.

Then $xz = 0^{2i}1^i \in L$.

And $yz = 0^{i+j}1^i \notin L$, because $i + j \neq 2i$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (reduction via closure): If L were regular, then the language

$$L \cap 0^*1^* = \left\{0^{2n}1^n \mid n \geq 0\right\}$$

would also be regular, because regular languages are closed under intersection. But we just proved that $\left\{0^{2n}1^n \mid n \geq 0\right\}$ is not regular in problem 2. ■

5. Strings of properly nested parentheses ( ), brackets [ ], and braces { }. For example, the string ([]){} is in this language, but the string ([)] is not, because the left and right delimiters don't match.

Solution (fooling set):

Let F be the language $(^*$.

Let x and y be arbitrary distinct strings in F.

Then $x = (^i$ and $y = (^j$ for some non-negative integers $i \neq j$.

Let $z =)^i$.

Then $xz = (^i)^i \in L$.

And $yz = (^j)^i \notin L$, because $i \neq j$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

[Notice that this argument doesn't even try to consider strings with different types of brackets, because it doesn't have to. We proved that L has an infinite fooling set F; that's enough.]

Solution (reduction via closure and renaming): If L were regular, then the language

$$L' := L \cap [^*]^* = \{[^n]^n \mid n \geq 0\}$$

would also be regular, because regular languages are closed under intersection. But L' is the same as the language $\{0^n1^n \mid n \geq 0\}$, except for renaming the symbols $0 \mapsto [$ and $1 \mapsto]$, and we proved that $\{0^n1^n \mid n \geq 0\}$ in class. ■

Harder problems to think about later:

6. Strings of the form $w_1\#w_2\#\cdots\#w_n$ for some $n \geq 2$, where each substring w_i is a string in $\{0, 1\}^*$, and some pair of substrings w_i and w_j are equal.

Solution (fooling set for the special case $n = 2$):

Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = \#0^i$.

Then $xz = 0^i\#0^i \in L$.

And $yz = 0^j\#0^i \notin L$, because $i \neq j$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Notice that this argument doesn't even try to consider strings with more than one #, because it doesn't have to. We proved that L has an infinite fooling set F; that's enough.

7. $\{w \in (0 + 1)^* \mid w = x1^n \text{ for some string } x \text{ with } |x| = n\}$, or less formally, binary strings whose right half contains only 1s.

Solution (Fooling set $F = 0^*$): Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Without loss of generality, assume $i < j$ (otherwise swap).

Let $z = 1^i$.

Then $xz = 0^i1^i \in L$.

Also $yz = 0^j1^i \notin L$. There are two cases to consider:

- If $|yz| = i + j$ is odd, then yz is not in L.
- Suppose $|yz| = i + j$ is even. The right half of yz has length $(i + j)/2 > i$ and thus contains at least one 0, so again yz is not in L.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

8. $\{\mathtt{0}^{n^2} \mid n \geq 0\}$

Solution (fooling set $F = L$): Let x and y be arbitrary distinct strings in L.

Without loss of generality, $x = \mathtt{0}^{i^2}$ and $y = \mathtt{0}^{j^2}$ for some $i > j \geq 0$.

Let $z = \mathtt{0}^{2j+1}$.

Then $xz = \mathtt{0}^{i^2+2j+1} \notin L$, because $i^2 < i^2 + 2j + 1 < (i+1)^2$.

On the other hand, $yz = \mathtt{0}^{j^2+2j+1} = \mathtt{0}^{(j+1)^2} \in L$

Thus, z distinguishes x and y.

We conclude that L is a fooling set for L.

Because L is infinite, L cannot be regular. ■

Solution (fooling set $F = \mathtt{0}^*$): Let x and y be arbitrary distinct strings in $\mathtt{0}^*$.

Without loss of generality, $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some $i > j \geq 0$.

Let $z = \mathtt{0}^{i^2+i+1}$.

Then $xz = \mathtt{0}^{i^2+2i+1} = \mathtt{0}^{(i+1)^2} \in L$.

On the other hand, $yz = \mathtt{0}^{i^2+i+j+1} \notin L$, because $i^2 < i^2 + i + j + 1 < (i+1)^2$.

Thus, z distinguishes x and y.

We conclude that $\mathtt{0}^*$ is a fooling set for L.

Because $\mathtt{0}^*$ is infinite, L cannot be regular. ■

Solution (fooling set $F = \mathtt{0000}^*$): Let x and y be arbitrary distinct strings in $\mathtt{0000}^*$.

Without loss of generality, $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some $i > j \geq 3$.

Let $z = \mathtt{0}^{i^2-i}$.

Then $xz = \mathtt{0}^{i^2} \in L$.

On the other hand, $yz = \mathtt{0}^{i^2-i+j} \notin L$, because

$$(i-1)^2 \;=\; i^2 - 2i + 1 \;<\; i^2 - i \;<\; i^2 - i + j \;<\; i^2.$$

(The first inequality requires $i \geq 2$, and the second requires $j \geq 1$.)

Thus, z distinguishes x and y.

We conclude that $\mathtt{0000}^*$ is a fooling set for L.

Because $\mathtt{0000}^*$ is infinite, L cannot be regular. ■

*9. $\{w \in (\mathtt{0}+\mathtt{1})^* \mid w$ is the binary representation of a perfect square$\}$

Solution (fooling set): We design an infinite fooling set around numbers of the form $(2^k+1)^2 = 2^{2k} + 2^{k+1} + 1 = \mathtt{1}\mathtt{0}^{k-2}\mathtt{1}\mathtt{0}^k\mathtt{1} \in L$, for any integer $k \geq 2$. The argument is somewhat simpler if we further restrict k to be even.

Let $F = \mathtt{1}(\mathtt{00})^*\mathtt{1}$, and let x and y be arbitrary distinct strings in F.

Then $x = \mathtt{1}\mathtt{0}^{2i-2}\mathtt{1}$ and $y = \mathtt{1}\mathtt{0}^{2j-2}\mathtt{1}$, for some positive integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let $z = \mathtt{0}^{2i}\mathtt{1}$.

Then $xz = \mathtt{1}\mathtt{0}^{2i-2}\mathtt{1}\mathtt{0}^{2i}\mathtt{1}$ is the binary representation of $2^{4i} + 2^{2i+1} + 1 = (2^{2i}+1)^2$, and therefore $xz \in L$.

On the other hand, $yz = \mathtt{1}\mathtt{0}^{2j-2}\mathtt{1}\mathtt{0}^{2i}\mathtt{1}$ is the binary representation of the integer $2^{2i+2j} + 2^{2i+1} + 1$. Simple algebra gives us the inequalities

$$\begin{aligned}
(2^{i+j})^2 &= 2^{2i+2j} \\
&< \mathbf{2^{2i+2j} + 2^{2i+1} + 1} \\
&< 2^{2(i+j)} + 2^{i+j+1} + 1 \\
&= (2^{i+j}+1)^2.
\end{aligned}$$

So $2^{2i+2j} + 2^{2i+1} + 1$ lies between two consecutive perfect squares, and thus is not a perfect square, which implies that $yz \notin L$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ∎