GPS 10: Mon Nov 16
HW 10: Tue Nov 17

Optimization problems: Given an input. There are many "feasible" solutions but one is best. Find it.

Ex: Given directed $G=(V,E)$ & $s,t \in V$, what is the shortest path from s to t?

☆ Decision problems: Given an input. Answer Yes or No.

Ex: Given directed $G=(V,E)$ & $s,t \in V$ & an integer k, is the distance from s to t at most k?

Focus on decision problems from here on.

Let A be a decision problem.

Treat input to A as a string to describe it.

The strings for Yes instances form a subset of all strings...

a language.

L_A: Set of inputs (as strings) for Yes instances of A.

vice versa: The decision problem for a language L is "Given string x, is $x \in L$?"

Suppose we have a decision problem A / its language L_A.

Many-one reduction from a decision problem A to decision problem B... is a function f (an algorithm) from instances of A to instances of B s.t.

x is Yes for A iff f(x) is Yes for B.

(have to be able to "compute" f, i.e. there is an algorithm to find f(x))

i.e.

$x \in L_A$ iff $f(x) \in L_B$

A <u>polynomial time many-one reduction</u>

(<u>Karp reduction</u>) is a many-one reduction that takes time polynomial in $|x|$

(size of input)

If one exists going from A to B,

we write $A \leq_p B$ or $L_A \leq_p L_B$.

less than or equal to symbol

"A is no more difficult than B up to

some polynomial overhead."

Lemma: If $A \leq_p B$ and B has a poly time algorithm, then A has a poly time algorithm.

Proof: If f take $O(|x|^c)$ time & the algo for B take $O(|y|^d)$ time on an input y. Can solve input x for A by solving $f(x)$ for B in

$$O(|x|^c + (|x|^c)^d) = O(|x|^{cd}) \text{ time.}$$

find ~~solve~~ $f(x)$

solve $f(x)$ for B

$f(x)$ cannot be larger

in $|x|^c$ time

Corollary: If $A \leq_p B$, and A has no polynomial time algorithm, then B has no polynomial time algorithm.

Ex: Three problems that take
- an undirected $G=(V,E)$
- an integer $k \geq 0$

Independent Set (IS):

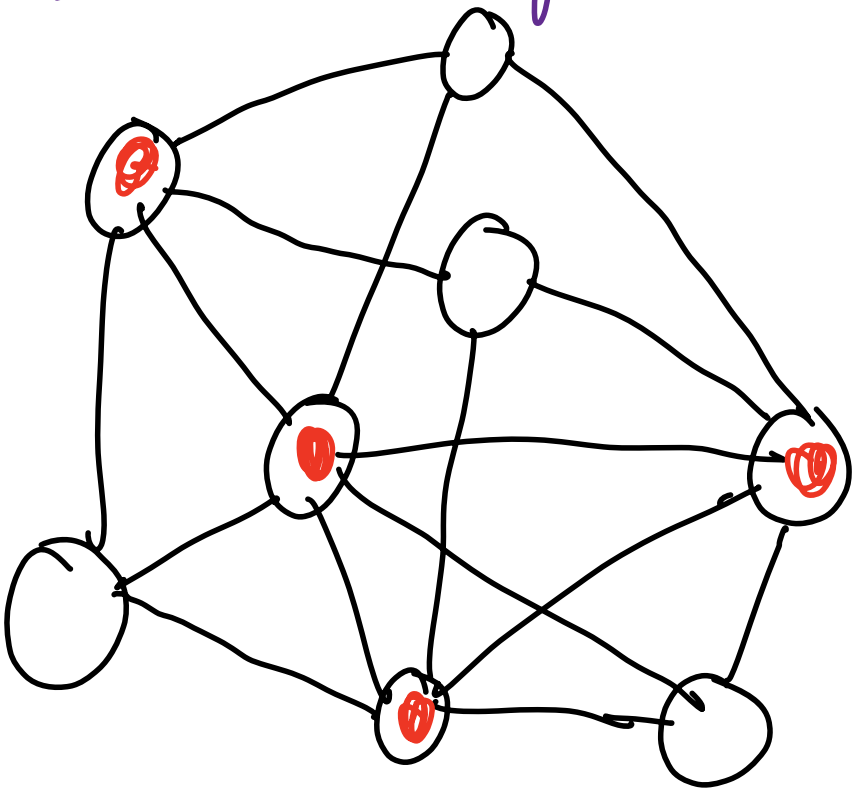
A ind. set S is a subset of V where no two members of S share an edge.

Is there an independent set of size k? (or greater)

Clique:

A clique S is a subset of V s.t. every pair of vertices in S share an edge.

Is there a clique of size k?



← Vertex cover of size 4.

Vertex Cover (VC)

A vertex cover S is a subset of V s.t. for all $uv \in E$, $u \in S$, $v \in S$, or both.

Is there a vertex cover of size k? (or less)

Theorem: $IS \leq_p$ Clique

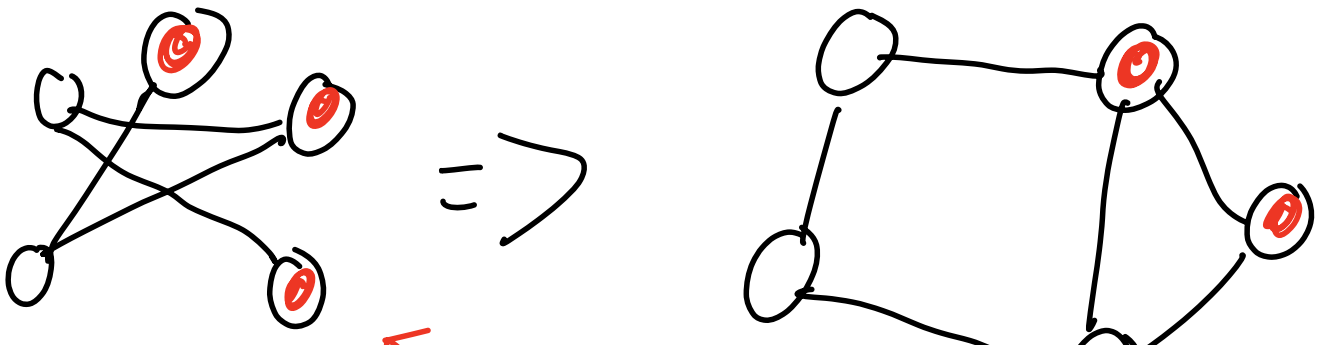
Given undirected $G=(V,E)$ & k as instance of IS.

Build instance of Clique $G'=(V', E')$ & k'.

$V' := V$

$k' := k$.

$E' := \{uv \mid uv \notin E\}$



IS Yes for $k=3$



Clique Yes for $k=3$.

Reduction runs in polynomial $O(|V|^2)$ time

Claim: G has an ind. set of size k iff G' has a clique of size k'.

Suppose $S \subseteq V$ is ind. in G and $|S| = k$.

For all $u, v \in S$, $uv \notin E$.

So $uv \in E'$.

So S is a clique in G' with $|S| = k = k'$.

Now suppose $S' \subseteq V'$ is a clique in G'.

For all $u, v \in S'$, $uv \in E'$.

So $uv \notin E$.

So S' is an ind. set in G with $|S'| = k' = k$.

(Clique $\leq_p$ IS by this reduction algo)

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Theorem: IS $\leq_p$ VC

(next time)