Prove that each of the following languages is ***not*** regular.

1. $\left\{0^{2^n} \mid n \geq 0\right\}$

Solution (fooling set $F = L$): Let $F = L = \left\{0^{2^n} \mid n \geq 0\right\}$.

Let x and y be arbitrary distinct elements of F.

Then $x = 0^{2^i}$ and $y = 0^{2^j}$ for some non-negative integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let $z = 0^{2^i}$.

- $xz = 0^{2^i}0^{2^i} = 0^{2^{i+1}} \in L$.
- $yz = 0^{2^j}0^{2^i} = 0^{2^i+2^j}$. The integer $2^i + 2^j$ lies strictly between 2^j and 2^{j+1} (because $i < j$) and thus is not a power of 2. It follows that $yz \notin L$.

Because $xz \in L$ and $yz \notin L$, the suffix z distinguishes x and y.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (fooling set $F = 0^*$): Let $F = 0^* = \{0^n \mid n \geq 0\}$.

Let x and y be arbitrary distinct elements of F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let r be any integer such that $2^r > j$, and let $z = 0^{2^r-i}$.

Then $xz = 0^i 0^{2^r-i} = 0^{2^r} \in L$.

But $yz = 0^j 0^{2^r-i} = 0^{2^r+j-i} \notin L$, because $2^r + j - i$ is not a power of 2:

$$
\begin{aligned}
2^r < \mathbf{2^r + j - i} & \qquad [i < j] \\
\leq 2^r + j & \qquad [i \geq 0] \\
< 2^r + 2^r & \qquad [j < 2^r] \\
= 2^{r+1} & \qquad [\text{math}]
\end{aligned}
$$

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (fooling set $F = 0^*$, but distinguish the other way):

Let $F = 0^* = \{0^n \mid n \geq 0\}$.

Let x and y be arbitrary distinct elements of F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let r be any integer such that $2^{r-1} > j$, and let $z = 0^{2^r-j}$.

Then $xz = \mathtt{0}^i\mathtt{0}^{2^r-j} = \mathtt{0}^{2^r-j+i} \notin L$, because $2^r - j + i$ is not a power of 2:

$$\begin{aligned} 2^{r-1} &= 2^r - 2^{r-1} && \text{[math]} \\ &< 2^r - j && [2^{r-1} > j] \\ &\le \boldsymbol{2^r - j + i} && [i \ge 0] \\ &< 2^r && [i < j] \end{aligned}$$

But $yz = \mathtt{0}^j\mathtt{0}^{2^r-j} = \mathtt{0}^{2^r} \in L$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

2. $\{0^{2n}1^n \mid n \geq 0\}$

Solution (fooling set $F = (00)^*$): Let F be the language $(00)^*$.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^{2i}$ and $y = 0^{2j}$ for some non-negative integers $i \neq j$.

Let $z = 1^i$.

Then $xz = 0^{2i}1^i \in L$.

And $yz = 0^{2j}1^i \notin L$, because $i \neq j$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (fooling set $F = 0^*$): Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i1^i$.

Then $xz = 0^{2i}1^i \in L$.

And $yz = 0^{i+j}1^i \notin L$, because $i + j \neq 2i$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (reduction via homomorphism): Suppose to the contrary that L is the language of some DFA $M = (Q, s, A, \delta)$. Construct a new DFA $M' = (Q, s, A, \delta')$ with the same states, start state, and accepting states as M, but with a new transition function:

$$\delta'(q, a) = \begin{cases} \delta^*(q, 00) & \text{if } a = 0 \\ \delta(q, 1) & \text{if } a = 1 \end{cases}$$

In other words, M' simulates M, but pretends that every 0 it reads is actually two 0s. Let *doubleoh* be the following string function:

$$doubleoh(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ 00 \bullet doubleoh(x) & \text{if } w = 0x \\ 1 \cdot doubleoh(x) & \text{if } w = 1x \end{cases}$$

In particular, for any integer n, we have $doubleoh(0^n1^n) = 0^{2n}1^n$. Straightforward but tedious induction implies that our new DFA M' accepts a string w if and only if the original DFA M accepts the string $doubleoh(w)$. It follows that $L(M') = \{0^n1^n \mid n \geq 0\}$. But we proved in class that $L(M')$ is not regular, so we have reached a contradiction; the original DFA M cannot exist! ■

3. $\{0^m 1^n \mid m \neq 2n\}$

Solution (fooling set $F = (00)^*$): Let F be the language $(00)^*$.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^{2i}$ and $y = 0^{2j}$ for some non-negative integers $i \neq j$.

Let $z = 1^i$.

Then $xz = 0^{2i}1^i \notin L$.

And $yz = 0^{2j}1^i \in L$, because $i \neq j$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (fooling set $F = 0^*$): Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i 1^i$.

Then $xz = 0^{2i}1^i \notin L$.

And $yz = 0^{i+j}1^i \in L$, because $i + j \neq 2i$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (reduction via closure): If L were regular, then the language

$$0^*1^* \setminus L \;=\; \{0^m 1^n \mid m = 2n\} \;=\; \left\{0^{2n}1^n \;\middle|\; n \geq 0\right\}$$

would also be regular, because regular languages are closed under complement. But we just proved that $\left\{0^{2n}1^n \;\middle|\; n \geq 0\right\}$ is not regular in problem 2. ■

4. Strings over $\{0, 1\}$ where the number of 0s is exactly twice the number of 1s.

Solution (fooling set $F = 1^*$): Let F be the language 1^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 1^i$ and $y = 1^j$ for some non-negative integers $i \neq j$.

Let $z = 0^{2i}$.

Then $xz = 1^i 0^{2i} \in L$.

And $yz = 1^j 0^{2i} \notin L$, because $i \neq j$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (fooling set $F = 0^*$): Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i 1^i$.

Then $xz = 0^{2i} 1^i \in L$.

And $yz = 0^{i+j} 1^i \notin L$, because $i + j \neq 2i$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (reduction via closure): If L were regular, then the language

$$L \cap 0^* 1^* \;=\; \left\{ 0^{2n} 1^n \;\middle|\; n \geq 0 \right\}$$

would also be regular, because regular languages are closed under intersection. But we just proved that $\left\{ 0^{2n} 1^n \;\middle|\; n \geq 0 \right\}$ is not regular in problem 2. ■

5. Strings of properly nested parentheses (), brackets [], and braces {}. For example, the string ([]){} is in this language, but the string ([)] is not, because the left and right delimiters don't match.

Solution (fooling set):

Let F be the language $(^*$.

Let x and y be arbitrary distinct strings in F.

Then $x = (^i$ and $y = (^j$ for some non-negative integers $i \neq j$.

Let $z =)^i$.

Then $xz = (^i)^i \in L$.

And $yz = (^j)^i \notin L$, because $i \neq j$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

[Notice that this argument doesn't even try to consider strings with different types of brackets, because it doesn't have to. We proved that L has an infinite fooling set F; that's enough.]

Solution (reduction via closure and renaming): If L were regular, then the language

$$L' := L \cap [^*]^* = \{[^n]^n \mid n \geq 0\}$$

would also be regular, because regular languages are closed under intersection. But L' is the same as the language $\{0^n1^n \mid n \geq 0\}$, except for renaming the symbols $0 \mapsto [$ and $1 \mapsto]$, and we proved that $\{0^n1^n \mid n \geq 0\}$ in class. ■

Harder problems to think about later:

6. Strings of the form $w_1\#w_2\#\cdots\#w_n$ for some $n \geq 2$, where each substring w_i is a string in $\{0,1\}^*$, and some pair of substrings w_i and w_j are equal.

Solution (fooling set for the special case $n = 2$):

Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = \#0^i$.

Then $xz = 0^i\#0^i \in L$.

And $yz = 0^j\#0^i \notin L$, because $i \neq j$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Notice that this argument doesn't even try to consider strings with more than one #, because it doesn't have to. We proved that L has an infinite fooling set F; that's enough.

7. $\{w \in (0+1)^* \mid w = x1^n \text{ for some string } x \text{ with } |x| = n\}$, or less formally, binary strings whose right half contains only 1s.

Solution (Fooling set $F = 0^*$): Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Without loss of generality, assume $i < j$ (otherwise swap).

Let $z = 1^i$.

Then $xz = 0^i1^i \in L$.

Also $yz = 0^j1^i \notin L$. There are two cases to consider:

- If $|yz| = i + j$ is odd, then yz is not in L.
- Suppose $|yz| = i + j$ is even. The right half of yz has length $(i + j)/2 > i$ and thus contains at least one 0, so again yz is not in L.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

8. $\{\texttt{0}^{n^2} \mid n \geq 0\}$

Solution (fooling set $F = L$): Let x and y be arbitrary distinct strings in L.

Without loss of generality, $x = \texttt{0}^{i^2}$ and $y = \texttt{0}^{j^2}$ for some $i > j \geq 0$.

Let $z = \texttt{0}^{2j+1}$.

Then $xz = \texttt{0}^{i^2+2j+1} \notin L$, because $i^2 < i^2 + 2j + 1 < (i+1)^2$.

On the other hand, $yz = \texttt{0}^{j^2+2j+1} = \texttt{0}^{(j+1)^2} \in L$

Thus, z distinguishes x and y.

We conclude that L is a fooling set for L.

Because L is infinite, L cannot be regular. ■

Solution (fooling set $F = \texttt{0}^*$): Let x and y be arbitrary distinct strings in $\texttt{0}^*$.

Without loss of generality, $x = \texttt{0}^i$ and $y = \texttt{0}^j$ for some $i > j \geq 0$.

Let $z = \texttt{0}^{i^2+i+1}$.

Then $xz = \texttt{0}^{i^2+2i+1} = \texttt{0}^{(i+1)^2} \in L$.

On the other hand, $yz = \texttt{0}^{i^2+i+j+1} \notin L$, because $i^2 < i^2 + i + j + 1 < (i+1)^2$.

Thus, z distinguishes x and y.

We conclude that $\texttt{0}^*$ is a fooling set for L.

Because $\texttt{0}^*$ is infinite, L cannot be regular. ■

Solution (fooling set $F = \texttt{0000}^*$): Let x and y be arbitrary distinct strings in $\texttt{0000}^*$.

Without loss of generality, $x = \texttt{0}^i$ and $y = \texttt{0}^j$ for some $i > j \geq 3$.

Let $z = \texttt{0}^{i^2-i}$.

Then $xz = \texttt{0}^{i^2} \in L$.

On the other hand, $yz = \texttt{0}^{i^2-i+j} \notin L$, because

$$(i-1)^2 \;=\; i^2 - 2i + 1 \;<\; i^2 - i \;<\; i^2 - i + j \;<\; i^2.$$

(The first inequality requires $i \geq 2$, and the second requires $j \geq 1$.)

Thus, z distinguishes x and y.

We conclude that $\texttt{0000}^*$ is a fooling set for L.

Because $\texttt{0000}^*$ is infinite, L cannot be regular. ■

*9. $\{w \in (0+1)^* \mid w \text{ is the binary representation of a perfect square}\}$

Solution (fooling set): We design an infinite fooling set around numbers of the form $(2^k+1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k1 \in L$, for any integer $k \geq 2$. The argument is somewhat simpler if we further restrict k to be even.

Let $F = 1(00)^*1$, and let x and y be arbitrary distinct strings in F.

Then $x = 10^{2i-2}1$ and $y = 10^{2j-2}1$, for some positive integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let $z = 0^{2i}1$.

Then $xz = 10^{2i-2}10^{2i}1$ is the binary representation of $2^{4i} + 2^{2i+1} + 1 = (2^{2i}+1)^2$, and therefore $xz \in L$.

On the other hand, $yz = 10^{2j-2}10^{2i}1$ is the binary representation of the integer $2^{2i+2j} + 2^{2i+1} + 1$. Simple algebra gives us the inequalities

$$\begin{aligned}(2^{i+j})^2 &= 2^{2i+2j} \\ &< \mathbf{2^{2i+2j} + 2^{2i+1} + 1} \\ &< 2^{2(i+j)} + 2^{i+j+1} + 1 \\ &= (2^{i+j}+1)^2.\end{aligned}$$

So $2^{2i+2j} + 2^{2i+1} + 1$ lies between two consecutive perfect squares, and thus is not a perfect square, which implies that $yz \notin L$.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■