Strings + induction

An arbitrary finite of **characters** or **symbols**:

an **alphabet**

ex. $\Sigma = \{A, B, C, \ldots, Z\}$

$\Sigma =$ cast of a webcomic

$\Sigma = \{0, 1\} \leftarrow$ typical

A <u>string</u> (also called a <u>word</u>) over Σ is a finite sequence of zero or more characters from Σ.

Formally a string w over Σ is defined recursively as one of the following

- the empty string, denoted ε
- an ordered pair (a, x) with $a \in \Sigma$
  x a string in Σ

$$\text{STRING} = (S, \text{TRING})$$

$$= (S, (T, \text{RING}))$$

$$\cdots$$

$$= (S, (T, (R, (I, (N, (G, \varepsilon))))))$$

(a, x) also written as $a \cdot x$

ax

Functions

<u>length</u> $|w|$ of a string w is the # characters in w.

$$|w| := \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + |x| & \text{if } w = ax \end{cases}$$

$$\begin{aligned} |HAT| &= 1 + |AT| \\ &= 1 + (1 + |T|) \\ &= 1 + (1 + (1 + |\varepsilon|)) \\ &= 1 + (1 + (1 + 0)) = 3 \end{aligned}$$

concatenation of strings w and z, denoted $w \bullet z$ (or wz) is all of w followed by all of z

$$w \bullet z := \begin{cases} z & \text{if } w = \varepsilon \\ a \cdot (x \bullet z) & \text{if } w = ax \end{cases}$$

$$\text{NOW} \bullet \text{HERE} = \text{N} \cdot (\text{OW} \bullet \text{HERE})$$

$$\vdots$$

$$= \text{NOWHERE}$$

Want to prove every string is "perfectly cromulent"

Proof: Let w be an arbitrary string.
Assume, for every string x such that $|x| < |w|$, that x is perfectly cromulent. ← strong induction hypothesis
There are two cases to consider.

- Suppose $w = \varepsilon$.

  Therefore, w is perfectly cromulent.

- Suppose $w = ax$ for some symbol a and string x.
  The induction hypothesis implies that x is perfectly cromulent.

  Therefore, w is perfectly cromulent.

In both cases, we conclude that w is perfectly cromulent. □

Lemma: For every string w we have $w \bullet \varepsilon = w$.

Proof: Let w be an arbitrary string.

Assume $x \bullet \varepsilon = x$ for every string x such that $|x| < |w|$.

There are two cases.

Suppose $w = \varepsilon$. Then

$$
\begin{aligned}
w \bullet \varepsilon &= \varepsilon \bullet \varepsilon && w = \varepsilon \\
&= \varepsilon && \text{def. } \bullet \\
&= w && w = \varepsilon
\end{aligned}
$$

Suppose $w = ax$ for some symbol a + string x.

$$
\begin{aligned}
w \bullet \varepsilon &= ax \bullet \varepsilon && w = ax \\
&= a \cdot (x \bullet \varepsilon) && \text{def. } \bullet \\
&= a \cdot x && \text{IH} \\
&= w && w = ax
\end{aligned}
$$

In all cases, $w \bullet \varepsilon = w$.

Lemma: $|w \bullet z| = |w| + |z|$ for all string w and z.

Proof: Let w and z be arbitrary strings.

Assume $|x \bullet z| = |x| + |z|$ for every string x such $|x| < |w|$.

There are two cases:

Suppose $w = \varepsilon$. Then,

$$
\begin{aligned}
|w \bullet z| &= |\varepsilon \bullet z| && w = \varepsilon \\
&= |z| && \text{def. } \bullet
\end{aligned}
$$

$$\begin{aligned}
&= 0 + |z| && \text{maths} \\
&= |e| + |z| && \text{def. } || \\
&= |w| + |z| && w = e
\end{aligned}$$

Suppose $w = ax$. Then,

$$\begin{aligned}
|w \bullet z| &= |ax \bullet z| && w = ax \\
&= |a \cdot (x \bullet z)| && \text{def. } \bullet \\
&= 1 + |x \bullet z| && \text{def. } || \\
&= 1 + |x| + |z| && \text{IH} \\
&= |ax| + |z| && \text{def. } || \\
&= |w| + |z| && w = ax
\end{aligned}$$

In all cases, $|w \bullet z| = |w| + |z|$.

Lemma: $(w \bullet y) \bullet z = w \bullet (y \bullet z)$

Proof: Let w, y, and z be arbitrary strings.

Assume that $(x \bullet y) \bullet z = x \bullet (y \bullet z)$ for every string x such $|x| < |w|$.

Two cases:

Suppose $w = \varepsilon$. Then

$$(w \bullet y) \bullet z = (\varepsilon \bullet y) \bullet z \qquad w = \varepsilon$$

$$= y \bullet z \qquad \text{def. } \bullet$$

$$= \varepsilon \bullet (y \bullet z) \qquad \text{def. } \bullet$$

$$= w \bullet (y \bullet z) \qquad w = \varepsilon$$

Suppose $w = ax$ for some char. a and string x.

$$(w \bullet y) \bullet z = (ax \bullet y) \bullet z \qquad w = ax$$

$$= (a \cdot (x \bullet y)) \bullet z \qquad \text{def. } \bullet$$

$$= a \cdot ((x \bullet y) \bullet z) \qquad \text{def. } \bullet$$

$$= a \cdot (x \bullet (y \bullet z)) \qquad \text{IH}$$

$$= ax \bullet (y \bullet z) \qquad \text{def. } \bullet$$

$$= w \bullet (y \bullet z) \qquad w = ax$$

In all cases, $(w \bullet y) \bullet z = w \bullet (y \bullet z)$.