A language (or its associated decision problem) was <u>acceptable</u> if there is a machine that for every input string in the language, the machine halts & accepts (say yes to) the string. (may reject or diverge otherwise)

A language is <u>decidable</u> if there is a machine that always halts & correctly reports accept or reject (Yes or No) otherwise.

<u>Undecidable</u> otherwise.

often use a description $\langle M \rangle$ of a machine M as input.

The canonical undecidable problem

$$\text{Halt} ::= \{ \langle M, w \rangle \mid M \text{ halts on } w \}$$

Accept, Reject, Diverge also undecidable.

not acceptable

To prove a language L is undecidable, reduce a known undecidable problem /language to L.

$\text{NeverHalt} := \{ \langle M \rangle \mid \text{Halt}(M) = \emptyset \}$

input strings that make M halt

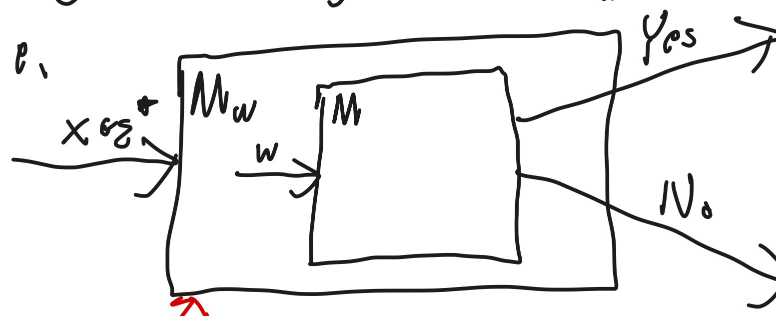
Thm: NeverHalt is undecidable.

Reduction from Halt.

Proof: Suppose there is a machine NH that decides NeverHalt.

Given instance $\langle M, w \rangle$ of Halt, let M_w be the following machine.

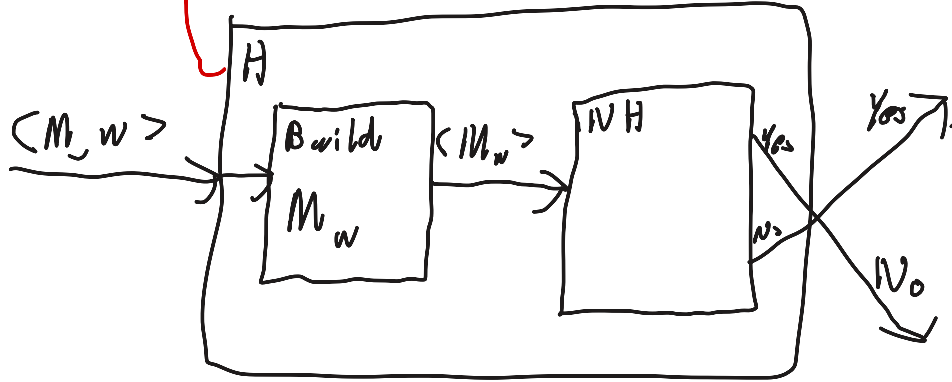
```
M_w(x):

return M(w)
```



Build a machine H to decide Halt.

$H(\langle M, w\rangle)$:

return $!NH(\langle M_w\rangle)$



Need to prove H decides Halt.

Suppose M halts on w.

M_w halts on every input x.

So NH rejects $\langle M_w\rangle$.

So H accepts $\langle M, w\rangle$.

Suppose M diverges on w.

M_w diverges on every input x. $\Rightarrow$ NH accepts $\langle M_w\rangle$

$\Rightarrow$ H rejects $\langle M, w\rangle$

So H decides Halt.

However, Halt is undecidable. $\perp$ NH cannot exist!

Never Accept, Never Reject, Never Diverge also undecidable.

$\text{HaltSame} := \{\langle M_1, M_2 \rangle \mid \text{Halt}(M_1) = \text{Halt}(M_2)\}$

Do M_1 & M_2 halt on the same set of strings?

Thm: HaltSame is undecidable.
(Reduce from NeverHalt)

Proof: Suppose there is a machine HS that decides HaltSame.

Let N be a machine that never halts (immediate infinite loop).

Build a machine NH for NeverHalt.

```
NH(<M>):

  return HS(<M, N>)
```

Need to prove NH decides NeverHalt.

Suppose M never halts on any input.

Neither does N, so HS accepts $\langle M, N \rangle$.

$\Rightarrow$ NH accepts $\langle M \rangle$.

Suppose M does halt on some input.

Machine N <u>never</u> halts, so HS reject $\langle M, N \rangle$.

$\Rightarrow$ NH reject $\langle M \rangle$.

NH does decide NeverHalt.
But NeverHalt is undecidable, so HS cannot exist.
(AcceptSome, etc. undecidable)

"Given $\langle M \rangle$, does M accept languages with
property?" ⟶

Rice's Theorem: Let $\mathcal{L}$ be any family of languages
(set of sets of strings) s.t.

- there is a machine Y s.t. $\text{Accept}(Y) \in \mathcal{L}$

all strings Y accepts

- there is a machine N s.t. $\text{Accept}(N) \notin \mathcal{L}$

Then,
$$\text{AcceptIn}(\mathcal{L}) := \{ \langle M \rangle \mid \text{Accept}(M) \in \mathcal{L} \}$$
is undecidable.

Proof: (Reduction from Halt)

Assume $\emptyset \notin \mathcal{L}$, and fix a machine Y s.t. $\text{Accept}(Y) \in \mathcal{L}$.

(symmetry for $\emptyset \in \mathcal{L}$).

Suppose there is a machine $A_{\mathcal{L}}$ that decides $\text{AcceptIn}(\mathcal{L})$.

Given an instance $\langle M, w \rangle$ of Halt, let M_w^+:

$M_w^+(x)$:

  $\_ \leftarrow M(w)$

  return $Y(x)$

$H(\langle M, w \rangle)$

  return $A_{\mathcal{L}}(\langle M_w^+ \rangle)$

Need to prove H decides Halt.

Suppose M halts on w.

For all strings x, $\mathrm{Accept}(M_w^+) = \mathrm{Accept}(Y) \in \mathcal{L}$

$\Rightarrow A_{\mathcal{L}}$ accepts $\langle M_w^+ \rangle$

$\Rightarrow H$ accepts $\langle M, w \rangle$

Suppose M does not halt on w.

Machine M_w^+ diverges on all x.

$\mathrm{Accept}(M_w^+) = \emptyset \notin \mathcal{L}$

$\Rightarrow A_{\mathcal{L}}$ rejects $\langle M_w^+ \rangle$

$\Rightarrow H$ reject $\langle M, w \rangle$

So H decides Halt. $\perp$. No machine $A_{\mathcal{L}}$.

Thm: The follow decision problems that take a machine description $\langle M \rangle$ are undecidable:

- Does M accept ε?
  - $\mathcal{L}$: all languages containing ε — Rice's thm
  - Y: accepts all strings (all ε)
  - N: accepts nothing
- Does M accept all palindromes with length 2^{prime}?
  - $\mathcal{L}$: all languages with all palindromes of length 2^{prime}
  - Y: accepts all strings
  - N: accepts nothing

- Does M accept a non-regular language?
  - R: all non-regular languages
  - Y: accepts the language $\{0^n 1^n \mid n \geq 0\}$
  - N: accepts all strings
    (Σ^* is not non-regular)