HW 9 due Tue Nov 4th
GP9 due Monday
Midterm 2 Mon Nov 10th

Given directed $G=(V,E)$
edge weights $w: E \to \mathbb{R}$
source $s \in V$.

v.dist : length of some $(s,v)$-path or $\infty$
v.pred : previous vertices on that path or Null

$u \to v$ is <u>tense</u> if $u.dist + w(u \to v) < v.dist$

<u>INITSSSP($s$):</u>
  $s.dist \leftarrow 0$
  $s.pred \leftarrow$ NULL
  for all vertices $v \neq s$
    $v.dist \leftarrow \infty$
    $v.pred \leftarrow$ NULL

<u>RELAX($u \to v$):</u>
  $v.dist \leftarrow u.dist + w(u \to v)$
  $v.pred \leftarrow u$

<u>FORDSSSP($s$):</u>
  INITSSSP($s$)
  while there is at least one tense edge
    RELAX any tense edge

Arbitrary non-negative weights: Dijkstra (a best-first search)

use priority queue

make sure v's key is v.dist

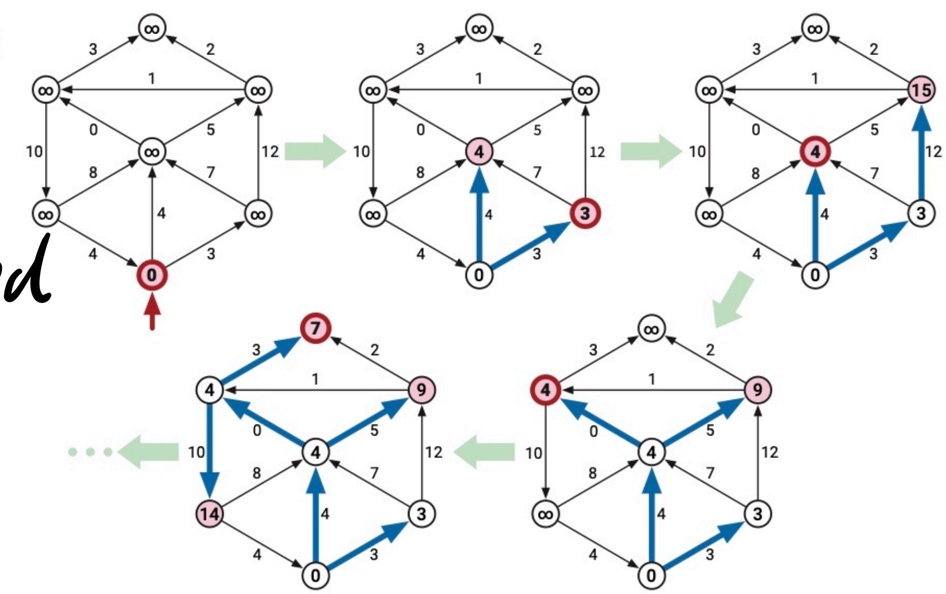
```
DIJKSTRA(s):
    INITSSSP(s)
    INSERT(s, 0)
    while the priority queue is not empty
        u ← EXTRACTMIN()
        for all edges u→v
            if u→v is tense
                RELAX(u→v)
                if v is in the priority queue
                    DECREASEKEY(v, v.dist)
                else
                    INSERT(v, v.dist)
```

← return vertex of min ~~key~~ priority

value

key

Is a Ford SSSP.



$d_i$: v.dist when v returned as the ith extraction

Claim: $d_{i+1} \ge d_i$.

$\Rightarrow$ after you extract v, it never gets extracted again

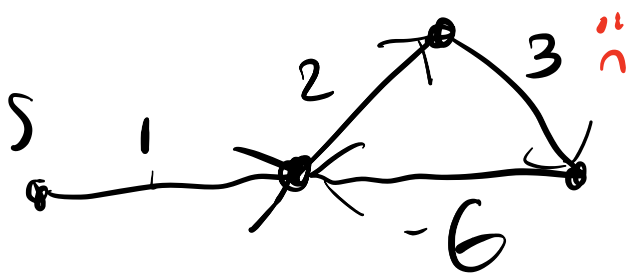
Time $O((V+E)\log V) = \underline{O(E \log V)}$ is G is connected

Negative weights:

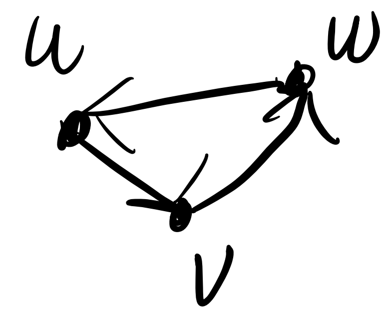
Dijkstra still correct... assuming no negative weight cycles. (otherwise, won't terminate)

Runs in $2^{\Theta(n)}$ time in worst case.

However, still fast in practise if very few negative weight edges (& no neg. cycles).



Lots of negative edges, but no negative cycle:




dist(v): distance from s to v

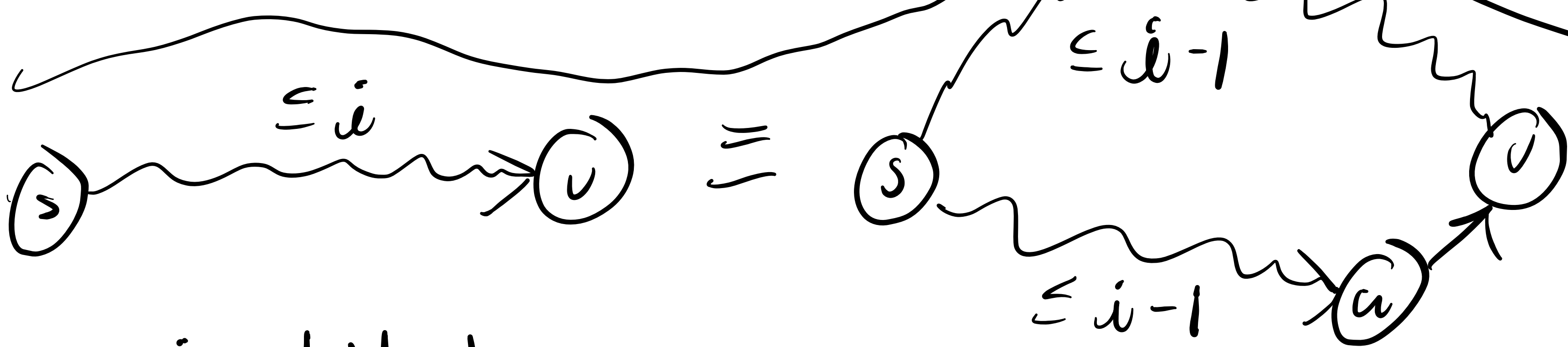
$$dist(v) = \begin{cases} 0 & \text{if } v = s \\ \min\limits_{u \to v} \left( dist(u) + w(u \to v) \right) & \text{otherwise} \end{cases}$$

$dist_{\leq i}(v)$: length of the shortest walk from s to v with at most i edges.

$$dist(v) = dist_{\leq |V|-1}(v)$$

$$dist_{\leq i}(v) = \begin{cases} 0 & \text{if } i = 0 \text{ \& } v = s \\ +\infty & \text{if } i = 0 \text{ \& } v \neq s \\ \min \left\{ \begin{array}{l} dist_{\leq i-1}(v) \\ \min\limits (dist_{\leq i-1}(u) + w(u \to v)) \end{array} \right\} & \text{otherwise} \end{cases}$$

$\min_{u \to v} \left( dist_{\le i-1}(u) + w \ldots \right)$




$0 \le i \le |V| - 1$

Eval in increasing order of $i$.

Store $dist[0..|V|-1, V]$

Running time:

$O(VE)$

```
BellmanFordDP(s)
    dist[0, s] ← 0
    for every vertex v ≠ s
        dist[0, v] ← ∞
    for i ← 1 to V − 1
        for every vertex v
            dist[i, v] ← dist[i − 1, v]
            for every edge u→v
                if dist[i, v] > dist[i − 1, u] + w(u→v)
                    dist[i, v] ← dist[i − 1, u] + w(u→v)
```

```
BellmanFordDP2(s)
  dist[0, s] ← 0
  for every vertex v ≠ s
      dist[0, v] ← ∞
  for i ← 1 to V − 1
      for every vertex v
          dist[i, v] ← dist[i − 1, v]
      for every edge u→v
          if dist[i, v] > dist[i − 1, u] + w(u→v)
              dist[i, v] ← dist[i − 1, u] + w(u→v)
```

```
BellmanFordDP3(s)
  dist[0, s] ← 0
  for every vertex v ≠ s
      dist[0, v] ← ∞
  for i ← 1 to V − 1
      for every vertex v
          dist[i, v] ← dist[i − 1, v]
      for every edge u→v
          if dist[i, v] > dist[i, u] + w(u→v)        ⟨⟨not i − 1!⟩⟩
              dist[i, v] ← dist[i, u] + w(u→v)    ⟨⟨not i − 1!⟩⟩
```

Claim: With DP3, $dist[i, v] \le dist_{\le i}(v)$

```
BellmanFordFinal(s)
  dist[s] ← 0
  for every vertex v ≠ s
      dist[v] ← ∞
  for i ← 1 to V − 1
      for every edge u→v
          if dist[v] > dist[u] + w(u→v)
              dist[v] ← dist[u] + w(u→v)
```

```
BellmanFord(s)
  InitSSSP(s)
  while there is at least one tense edge
      for every edge u→v
          if u→v is tense
              Relax(u→v)
```

Still $O(VE)$, Maybe faster.

What if there is a negative cycle?

```
BellmanFord(s)
    InitSSSP(s)
    repeat V − 1 times
        for every edge u→v
            if u→v is tense
                Relax(u→v)
    for every edge u→v
        if u→v is tense
            return "Negative cycle!"
```

Summary: dag: DagSSSP in $O(V+E)$

unweighted: BFS in $O(V+E)$

no (or few) negative edges: Dijkstra in $O(E \log V)$

o.w: Bellman-Ford in $O(VE)$

NEW: Non-negative weights $O(E \log^{2/3} V)$ [DMMSY '25]

Negative weights:

$$\tilde{\partial}\left(EV^{3/4}+E^{4/5}V\right)$$

[HJ<u>Q</u> '26]