Regular languages:

DFA

NFA

Regular expression

If L_1 and L_2 are regular, these are regular:

- $L_1 \cup L_2$
- $L_1 \cdot L_2$
- $(L_1)^*$
- $L_1 \cap L_2$
- $\overline{L_1} = \Sigma^* \setminus L_1$

Say we have a function f that takes a language L and returns another language.

Want to show if L is regular, then $f(L)$ is regular.

How: 1) Describe an algorithm that takes a regular expression R and outputs a regular expression R' s.t. $L(R') = f(L(R))$.

2) Describe an algorithm that takes a DFA M and outputs an NFA M' s.t. $L(M') = f(L(M))$.

$$w^R := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \cdot a & \text{if } w = ax \end{cases}$$

$$L^R = \{ w^R \mid w \in L \}.$$

<u>Lemma</u>: If L is regular, then L^R is regular.

<u>Proof</u> (via regular expressions):

Let L be a regular language.

There is a regular expression R s.t. $L = L(R)$.

Assume for any proper subexpression S of R that $L(S)^R$ is regular and has a regular expression S^R.

There are five cases:

$R = \emptyset$: Then, let $R^R = R$. $L^R = L = \emptyset$, so

$$L(R^R) = L(R) = L^R$$

$R = w$ for a string w: $R^R = w^R$

$R = A + B$. $R^R = A^R + B^R$

exist by IH

$$L(R^R) = L(A^R) \cup L(B^R) = L(A)^R \cup L(B)^R = L(A+B)^R$$

Suppose $R = AB$. Let $R^R = B^R A^R$

Suppose $R = A^*$. Let $R^R = (A^R)^*$

In all cases we found a regular expression R^R s.t. $L(R^R) = L(R)^R$.

Therefore $L(R)^R$ is regular.

Proof (via NFA construction).

Let $M = (\Sigma, Q, s, A, \delta)$ be an arbitrary DFA that accepts L.

We'll construct an NFA with multiple start states $M^R = (\Sigma, Q^R, S^R, A^R, \delta^R)$ that accepts L^R.

Want to "reverse the transitions".

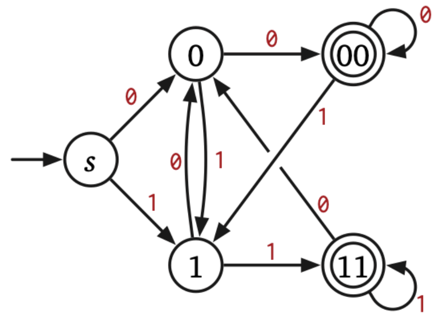
M^R:

$Q^R = Q$

$s^R = A$

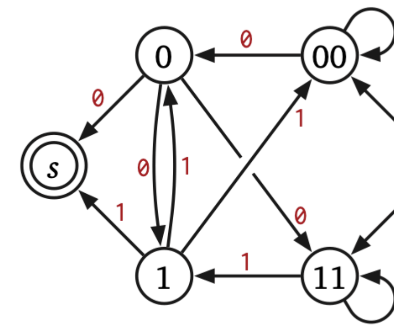
$A^R = \{s\}$

DFA M



$\Rightarrow$

NFA M^R



0 1 0 11

11 0 1 0

$\delta^R(q, a) = \{p \mid \delta(p, a) = q\}$ for all $q \in Q$
all $a \in \Sigma$

$\text{palin}(L) := \{ w \mid w \bullet w^R \in L \}$: the first half of even-length palindromes in L.

If $\underbrace{01101}\underbrace{10110} \in L$, then $01101 \in \text{palin}(L)$

Lemma: If L is regular, then palin(L) is regular.

<u>Proof:</u> Let $M = (\Sigma, Q, s, A, \delta)$ be an arbitrary DFA that accepts L.

M:

$$s \xrightarrow{01101} h \xrightarrow{10110} r \in A$$

$$s \xrightarrow[M]{01101} h$$

$$r \in A \xrightarrow[01101]{M^R} h$$

Build an NFA $M' = (\Sigma, Q', S', A', \delta')$

$$Q' = Q \times Q \qquad (q, r)$$

$$S' = \{(s, r) \mid r \in A\}$$

$$A' = \{(h, h) \mid h \in Q\}$$

$$\delta'((q, r), a) = \{(\delta(q, a), p) \mid \delta(p, a) = r\}$$

$\text{rev-suffix}(L) = \{xy \mid xy^R \in L\}$.

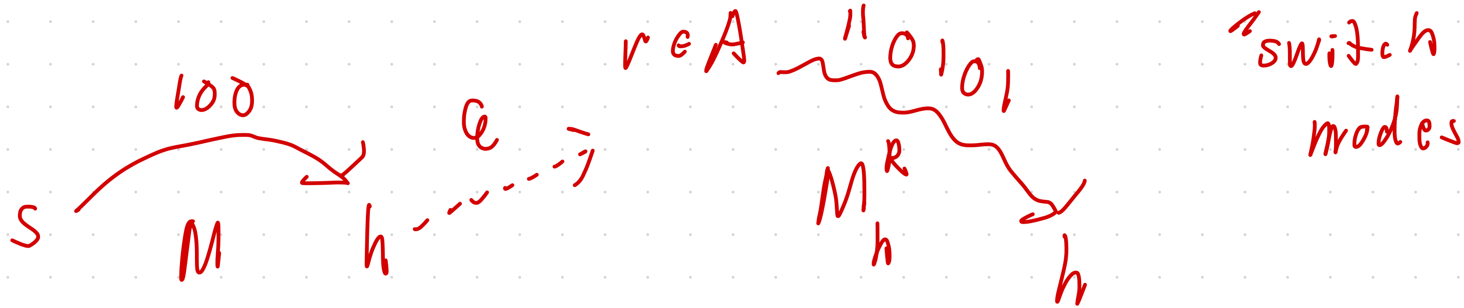
If $100101011 = (100 \cdot 101011) \in L$,

then $(100 \cdot 110101) = 100110101 \in \text{rev-suffix}(L)$

Lemma: If L is regular, $\text{rev-suffix}(L)$ is regular.

Proof: Let $M = (\Sigma, Q, s, A, \delta)$ be an arbitrary DFA that accepts L...

$$s \xrightarrow{100} h \xrightarrow{101011} r \in A$$



Make an NFA M' with ε-transitions.

target of M^R

$$Q' = Q \sqcup (Q \times Q)$$

current state of M^R

$$S' = \{s\}$$

$$A' = \{(h, h) \mid h \in Q\}$$

$\delta'(q, a) = \{\delta(q, a)\}$ for all $q \in Q$, $a \in \Sigma$

$\delta'(q, \epsilon) = \{(q, r) \mid r \in A\}$ for all $q \in Q$

time to read y?

$\delta'((h, q), a) = \{(h, p) \mid \delta(p, a) = q\}$

go backwards while checking y

$\delta'((h, q), \epsilon) = \emptyset$