A ***subsequence*** of a sequence (for example, an array, linked list, or string), obtained by removing zero or more elements and keeping the rest in the same order. A subsequence is called a ***substring*** or an ***interval*** if its elements are contiguous in the original sequence. For example:

- SUBSEQUENCE, UBSEQU, and the empty string ε are all substrings (and therefore subsequences) of the string SUBSEQUENCE;
- SBSQNC, SQUEE, and EEE are all subsequences of SUBSEQUENCE but not substrings;
- QUEUE, EQUUS, and DIMAGGIO are not subsequences (and therefore not substrings) of SUBSEQUENCE.

Questions to ask while designing a recursive backtracking algorithm:

- **How you would systematically solve the problem yourself?** By hand, not by writing code. What sequence of decisions do you need to make? For each decision, what are your options?
- **After you've made some decisions, what portion(s) of the input data is(are) left?** Equivalently, what is the "shape" of your recursive subproblems? Prefixes, suffixes, intervals, or something else? Do you need to remember anything else about your past decisions to make consistent future decisions?
- **How can you specify the "shape" of a subproblem with only a small number of parameters?** (typically indices into an input array)?
- **What precise recursive problem are you trying to solve?** Equivalently, after making some decisions, what question are you asking about the remaining input data? ***This is the most important step.***
- Finally, develop a recursive algorithm that answers the precise question you specified in the previous step. Your recursive algorithm should explore all possible options for exactly one decision, and each decision should be evaluated by recursively solving any resulting subproblem(s).

Describe ***recursive backtracking*** algorithms for the following longest-subsequence problems. *Don't worry about running times.*

1. Given an array $A[1..n]$ of integers, compute the length of a longest ***increasing*** subsequence. A sequence $B[1..\ell]$ is *increasing* if $B[i] > B[i-1]$ for every index $i \geq 2$.

   For example, given the array

   $$\langle 3, \underline{1}, \underline{4}, 1, \underline{5}, 9, 2, \underline{6}, 5, 3, 5, \underline{8}, 9, 7, \underline{9}, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$$

   your algorithm should return the integer 6, because $\langle 1, 4, 5, 6, 8, 9 \rangle$ is a longest increasing subsequence (one of many).

2. Given an array $A[1..n]$ of integers, compute the length of a longest ***decreasing*** subsequence. A sequence $B[1..\ell]$ is *decreasing* if $B[i] < B[i-1]$ for every index $i \geq 2$.

   For example, given the array

   $$\langle 3, 1, 4, 1, 5, \underline{9}, 2, \underline{6}, 5, 3, \underline{5}, 8, 9, 7, 9, 3, 2, 3, 8, \underline{4}, 6, \underline{2}, 7 \rangle$$

   your algorithm should return the integer 5, because $\langle 9, 6, 5, 4, 2 \rangle$ is a longest decreasing subsequence (one of many).

3. Given an array $A[1..n]$ of integers, compute the length of a longest ***alternating*** subsequence. A sequence $B[1..\ell]$ is *alternating* if $B[i] < B[i-1]$ for every even index $i \geq 2$, and $B[i] > B[i-1]$ for every odd index $i \geq 3$.

   For example, given the array

   $$\langle \underline{3}, \underline{1}, \underline{4}, \underline{1}, \underline{5}, 9, \underline{2}, \underline{6}, \underline{5}, 3, 5, \underline{8}, 9, \underline{7}, \underline{9}, \underline{3}, 2, 3, \underline{8}, \underline{4}, \underline{6}, \underline{2}, \underline{7} \rangle$$

   your algorithm should return the integer 17, because $\langle 3, 1, 4, 1, 5, 2, 6, 5, 8, 7, 9, 3, 8, 4, 6, 2, 7 \rangle$ is a longest alternating subsequence (one of many).

Harder problems to think about later:

4. Given an array $A[1..n]$ of integers, compute the length of a longest ***convex*** subsequence of A. A sequence $B[1..\ell]$ is *convex* if $B[i] - B[i-1] > B[i-1] - B[i-2]$ for every index $i \geq 3$.

   For example, given the array

   $$\langle \underline{3}, \underline{1}, 4, \underline{1}, 5, 9, \underline{2}, 6, 5, 3, \underline{5}, 8, \underline{9}, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$$

   your algorithm should return the integer 6, because $\langle 3, 1, 1, 2, 5, 9 \rangle$ is a longest convex subsequence (one of many).

5. Given an array $A[1..n]$, compute the length of a longest ***palindrome*** subsequence of A. Recall that a sequence $B[1..\ell]$ is a *palindrome* if $B[i] = B[\ell - i + 1]$ for every index i.

   For example, given the array

   $$\langle 3, 1, \underline{4}, 1, 5, \underline{9}, 2, 6, \underline{5}, \underline{3}, \underline{5}, 8, 9, 7, \underline{9}, 3, 2, 3, 8, \underline{4}, 6, 2, 7 \rangle$$

   your algorithm should return the integer 7, because $\langle 4, 9, 5, 3, 5, 9, 4 \rangle$ is a longest palindrome subsequence (one of many).