

Prove that each of the following languages is **not** regular, first using fooling sets and then (for problems 3, 4, and 5) using a reduction argument. You may use the fact (proven in class and in the lecture notes) that the language $\{0^n 1^n \mid n \geq 0\}$ is not regular. See the next page for a solved example showing both types of proof.

1. $\{0^{2^n} \mid n \geq 0\}$

2. $\{0^{2^n} 1^n \mid n \geq 0\}$

3. $\{0^m 1^n \mid m \neq 2n\}$

[Hint: There is a short reduction argument, but write the fooling set argument first.]

4. Strings over $\{0, 1\}$ where the number of 0s is exactly twice the number of 1s.

[Hint: There is a short reduction argument, but write the fooling set argument first.]

5. Strings of properly nested parentheses $()$, brackets $[]$, and braces $\{\}$. For example, the string $([])\{\}$ is in this language, but the string $([])]$ is not, because the left and right delimiters don't match.

[Hint: There is a short reduction argument, but write the fooling set argument first.]

Harder problems to think about later:

6. Strings of the form $w_1 \# w_2 \# \dots \# w_n$ for some $n \geq 2$, where each substring w_i is a string in $\{0, 1\}^*$, and some pair of substrings w_i and w_j are equal.

7. $\{w \in (0 + 1)^* \mid w = x 1^n \text{ for some string } x \text{ with } |x| = n\}$, or less formally, even-length binary strings whose right half contains only 1s.

8. $\{0^{n^2} \mid n \geq 0\}$

9. $\{w \in (0 + 1)^ \mid w \text{ is the binary representation of a perfect square}\}$

Solved problem:

10. Prove that the language $L = \{w \in (\emptyset + 1)^* \mid \#(\emptyset, w) = \#(1, w)\}$ is **not** regular.

Solution (fooling set $\emptyset^*$):

Consider the infinite set $F = \{\emptyset^n \mid n \geq 0\}$, or more simply $F = \emptyset^*$.

We claim that every pair of distinct strings in F has a distinguishing suffix.

Let x and y be arbitrary distinct strings in F .

The definition of F implies $x = \emptyset^i$ and $y = \emptyset^j$ for some integers $i \neq j$.

Let z be the string 1^i .

Then $xz = \emptyset^i 1^i \in L$.

But $yz = \emptyset^j 1^i \notin L$, because $i \neq j$.

So z is a distinguishing suffix for x and y .

We conclude that F is a fooling set for L .

Because F is infinite, L cannot be regular. ■

*This is **exactly** the proof from the lecture notes for the canonical non-regular language $\{\emptyset^n 1^n \mid n \geq 0\}$. The inner box is a proof that every pair of distinct strings in F has a distinguishing suffix.*

Solution (fooling set $\emptyset^*$):

For any natural number n , let $x_n = \emptyset^n$, and let $F = \{x_n \mid n \geq 0\} = \emptyset^*$.

Let i and j be arbitrary distinct natural numbers.

Let z_{ij} be the string 1^i .

Then $x_i z_{ij} = \emptyset^i 1^i \in L$.

But $x_j z_{ij} = \emptyset^j 1^i \notin L$, because $i \neq j$.

So z_{ij} is a distinguishing suffix for x_i and x_j .

We conclude that F is a fooling set for L .

Because F is infinite, L cannot be regular. ■

*This is another way of writing exactly the same proof that emphasizes the counter intuition; any algorithm that recognizes L **must** count $\emptyset$ s.*

Solution (reduction via closure): For the sake of argument, suppose L is regular.

Then the language $L \cap \emptyset^* 1^* = \{\emptyset^n 1^n \mid n \geq 0\}$ would also be regular, because regular languages are closed under intersection.

But we proved in class that $\{\emptyset^n 1^n \mid n \geq 0\}$ is not regular; we've reached a contradiction.

We conclude that L cannot be regular.

*And this is **why** the proof for $\{\emptyset^n 1^n \mid n \geq 0\}$ also works verbatim for this language. ■*