**Write your answers in the separate answer booklet.**
You have 180 minutes (after you get the answer booklet) to answer seven questions.

1. Recall that a ***run*** in a string $w \in \{0, 1\}^*$ is a maximal substring of $w$ whose characters are all equal. For example, the string 00011111110000 is the concatenation of three runs:

$$00011111110000 = 000 \bullet 1111111 \bullet 0000$$

   (a) Let $L_a$ denote the set of all non-empty strings in $\{0, 1\}^*$ where the length of the first run is equal to the number of runs. For example, $L_a$ contains the strings 0 and 1100000 and 0001110, but does not contain 000111 or 100011 or the empty string $\varepsilon$ (because it has no first run).

   ***Prove*** that $L_a$ is not a regular language.

   (b) Let $L_b$ denote the set of all strings in $\{0, 1\}^*$ that contain an even number of odd-length runs. For example, $L_b$ contains the strings 010111 and 1111 and the empty string $\varepsilon$, but does not contain either 0011100 or 11110.

   - Describe a DFA or NFA that accepts $L_b$ ***and***
   - Give a regular expression that describes $L_b$.

   (You do not need to prove that your answers are correct.)

2. Aladdin and Badroulbadour are playing a cooperative game. Each player has an array of positive integers, arranged in a row of squares from left to right. Each player has a token, which starts at the leftmost square of their row; their goal is to move *both* tokens onto the rightmost squares at the same time.

   On each turn, *both* players move their tokens *in the same direction*, either left or right. The distance each token travels is equal to the number under that token at the beginning of the turn. For example, if a token starts on a square labeled 5, then it moves either five squares to the right or five squares to the left. If *either* token moves past either end of its row, then both players immediately lose.

   For example, if Aladdin and Badroulbadour are given the arrays

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A$: | 7 | 5 | 4 | 1 | 2 | 3 | 3 | 2 | 3 | 1 | 4 | 2 |
| $B$: | 5 | 1 | 2 | 4 | 7 | 3 | 5 | 2 | 4 | 6 | 3 | 1 |

   they can win the game by moving right, left, left, right, right, left, right. On the other hand, if they are given the arrays

| | | | | | |
|---|---|---|---|---|---|
| $A$: | 2 | 3 | 5 | 1 | 3 |
| $B$: | 3 | 4 | 1 | 2 | 1 |

   they cannot win the game. (The first move must be to the right; then Aladdin's token moves out of bounds on the second turn.)

   Describe and analyze an algorithm to determine whether Aladdin and Badroulbadour can solve their puzzle, given the input arrays $A[1..n]$ and $B[1..n]$.

*Problems 3–7 appear on the following pages.*

3. Submit a solution to ***exactly one*** of the following problems. Don't forget to tell us which problem you've chosen!

   (a) Let $G = (V, E)$ be an arbitrary undirected graph. A subset $S \subseteq V$ of vertices is *mostly independent* if more than half the vertices of $S$ have no neighbors in $S$. ***Prove*** that finding the largest mostly independent set in $G$ is NP-hard.

   (b) ***Prove*** that the following problem is NP-hard: Given an undirected graph $G$, find the largest integer $k$ such that $G$ contains *two disjoint* independent sets of size $k$.

   (In fact, both of these problems are NP-hard, but we only want a proof for one of them.)

4. Recall that a *palindrome* is any string that is equal to its reversal, like `REDIVIDER` or `POOP`.

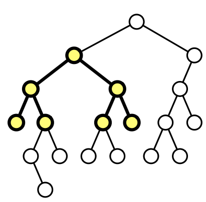
   (a) Describe and analyze an algorithm to find the length of the longest subsequence of a given string that is a palindrome.

   (b) A *double palindrome* is the concatenation of two *non-empty* palindromes, like `REFEREE` = `REFER` • `EE` or `POOPREDIVIDER` = `POOP` • `REDIVIDER`. Describe and analyze an algorithm to find the length of the longest subsequence of a given string that is a *double* palindrome. *[Hint: Use your algorithm from part (a).]*

   For both algorithms, the input is an array $A[1..n]$, and the output is an integer. For example, given the input string `MAYBEDYNAMICPROGRAMMING`, your algorithm for part (a) should return 7 (for the subsequences `NMRORMN` and `MAYBYAM`, among others), and your algorithm for part (b) should return 12 (for the subsequence `MAYBYAMIRORI`).

5. Recall that the *depth* of a vertex $v$ in a binary tree $T$ is the length of the unique path in $T$ from $v$ to the root of $T$. A binary tree is *complete* if every internal node has two children, and every leaf has exactly the same depth. An *internal subtree* of a binary tree $T$ is any connected subgraph of $T$.

   Describe and analyze a recursive algorithm to compute the *largest complete internal subtree* of a given binary tree. Your algorithm should return both the root and the depth of this internal subtree.



The largest complete internal subtree of this binary tree has depth 2.

*Problems 6 and 7 appear on the following pages.*

6. For each statement below, there are two boxes in the answer booklet labeled "Yes" and "No". Check "Yes" if the statement is ***always*** true and "No" otherwise, and give a ***brief*** (at most one short sentence) explanation of your answer. **Assume *P ≠ NP*.** If there is any other ambiguity or uncertainty about an answer, check "No". For example:

   - $x + y = 5$

     [X] Yes [ ] No — Suppose $x = 3$ and $y = 4$.

   - 3SAT can be solved in polynomial time.

     [ ] Yes [X] No — 3SAT is NP-hard.

   - If P = NP then Jeff is the Queen of England.

     [X] Yes [ ] No — The hypothesis is false, so the implication is true.

   Read each statement *very* carefully; some of these are deliberately subtle!

   (a) Which of the following statements are true?

   - The solution to the recurrence $T(n) = 4T(n/4) + O(n)$ is $T(n) = O(n \log n)$.
   - The solution to the recurrence $T(n) = 4T(n/4) + O(n^2)$ is $T(n) = O(n^2 \log n)$.
   - Every directed acyclic graph contains at most one source and at most one sink.
   - Depth-first search explores every path from the source vertex $s$ to every other vertex in the input graph.
   - Suppose $A[1..n]$ is an array of integers. Consider the following recursive function:

   $$Huh(i,j) = \begin{cases} 0 & \text{if } i < 0 \text{ or } j > n \\ \max\left\{ \begin{array}{c} Huh(i, j+1) \\ Huh(i-1, j) \\ A[i] \cdot A[j] + Huh(i-1, j+1) \end{array} \right\} & \text{otherwise} \end{cases}$$

     We can compute $Huh(n, 0)$ by memoizing this function into an array $Huh[0..n, 0..n]$ in $O(n^2)$ time, increasing $i$ in the outer loop and increasing $j$ in the inner loop.

   (b) Suppose we want to prove that the following language is undecidable.

   $$\text{Duck} := \left\{ \langle M \rangle \;\middle|\; M \text{ accepts } \texttt{GRAPES} \text{ but rejects } \texttt{LEMONADE} \right\}$$

   Professor Canard, your wetlands-ornithology instructor, suggests a reduction from the standard halting language

   $$\text{Halt} := \left\{ (\langle M \rangle, w) \;\middle|\; M \text{ halts on input } w \right\}.$$

   Specifically, suppose there is a Turing machine LooksLikeADuck that decides Duck. Professor Canard claims that the following algorithm decides Halt.

```
DecideHalt(⟨M⟩, w):
  Write code for the following algorithm:
      WaddleAway(x):
        run M on input w
        ⟨⟨ignore the output of M⟩⟩
        if x = LEMONADE
            return False
        else
            return True
  return LooksLikeADuck(⟨WaddleAway⟩)
```

Which of the following statements ***must be*** true ***for all*** inputs $\langle M\rangle\#w$?

- If $M$ accepts $w$, then WaddleAway accepts GRAPES.
- If $M$ diverges on $w$, then WaddleAway rejects GRAPES.
- If $M$ accepts $w$, then LooksLikeADuck accepts ⟨WaddleAway⟩.
- If $M$ diverges on $w$, then DecideHalt rejects $(\langle M\rangle, w)$.
- DecideHalt decides the language Halt. (That is, Professor Canard's reduction is correct.)

7. **More of the same:** For each statement below, there are two boxes in the answer booklet labeled "Yes" and "No". Check "Yes" if the statement is ***always*** true and "No" otherwise, and give a ***brief*** (at most one short sentence) explanation of your answer. **Assume *P ≠ NP*.**

   (a) Which of the following statements are true for ***all*** languages $L \subseteq \{0,1\}^*$?

   - $L^* = (L^*)^*$
   - If $L$ is decidable, then $L^*$ is decidable.
   - $L$ is either regular or NP-hard.
   - If $L$ is undecidable, then $L$ has an infinite fooling set.
   - The language $\{\langle M\rangle \mid M \text{ decides } L\}$ is undecidable.

   (b) Suppose there is a ***polynomial-time*** reduction from some language $A \subseteq \{0,1\}$ reduces to some other language $B \subseteq \{0,1\}$. Which of the following statements are true, assuming P ≠ NP?

   - $A \cap B \neq \varnothing$.
   - There is an algorithm to transform any Python program that solves $B$ in polynomial time into a Python program that solves $A$ in polynomial time.
   - If $B$ is NP-hard, then $A$ is NP-hard.
   - If $B$ is decidable, then $A$ is decidable.
   - If a Turing machine $M$ accepts every string in $B$, the *same* Turing machine $M$ also accepts every string in $A$.