Introduction to Dynamic Programming

Lecture 13

Thursday, October 10, 2024

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13.1
Recursion and Memoization

13.1.1
Fibonacci Numbers

Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

These numbers have many interesting properties. A journal The Fibonacci Quarterly!

1. **Binet's formula**: $F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} \approx \frac{1.618^n - (-0.618)^n}{\sqrt{5}} \approx \frac{1.618^n}{\sqrt{5}}$

   φ is the golden ratio $(1+\sqrt{5})/2 \simeq 1.618$.
2. $\lim_{n\to\infty} F(n+1)/F(n) = \varphi$

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How many bits?

Consider the nth Fibonacci number $F(n)$. Writing the number $F(n)$ in base 2 requires

(A) $\Theta(n^2)$ bits.
(B) $\Theta(n)$ bits.
(C) $\Theta(\log n)$ bits.
(D) $\Theta(\log \log n)$ bits.

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n − 1) + Fib(n − 2)
```

Running time? Let $T(n)$ be the number of additions in Fib(n).

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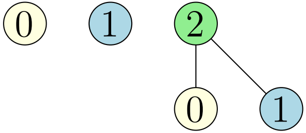
$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

Roughly same as $F(n)$: $T(n) = \Theta(\varphi^n)$.
The number of additions is exponential in n. Can we do better?

Recursion tree for the Recursive Fibonacci

Recursion tree for the Recursive Fibonacci



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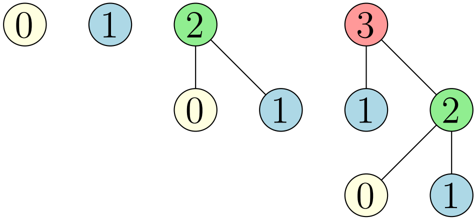

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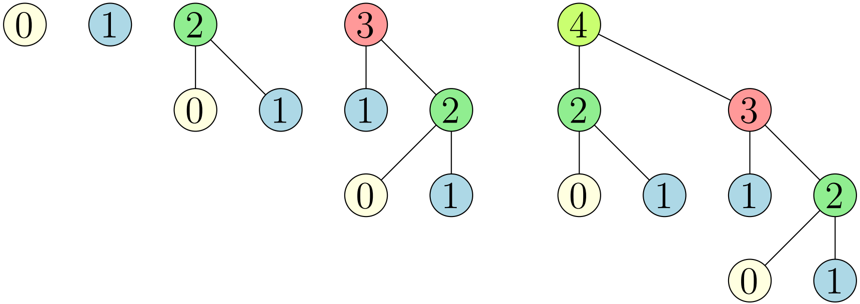

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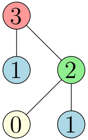

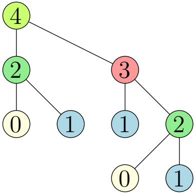

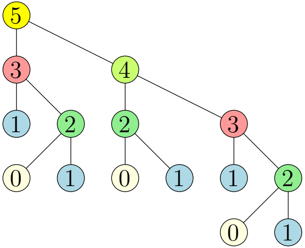

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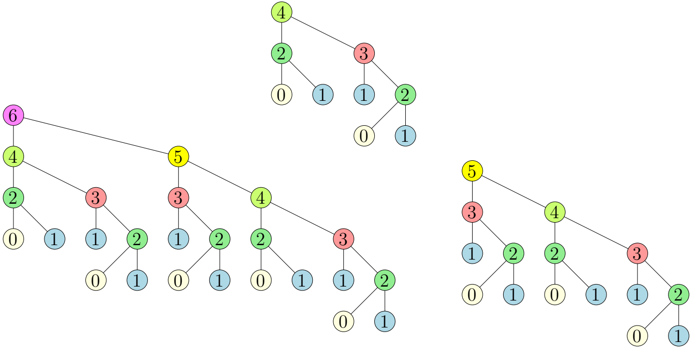

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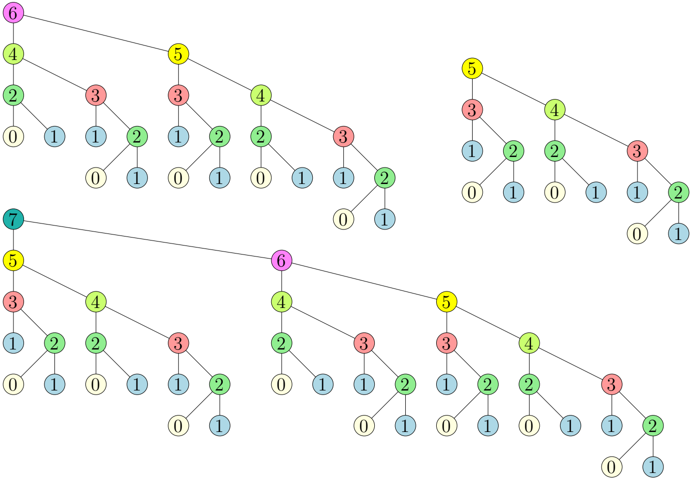

An iterative algorithm for Fibonacci numbers

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

What is the running time of the algorithm? $O(n)$ additions.

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What is the running time of the algorithm? $O(n)$ additions.

What is the difference?

1. Recursive algorithm is computing the same numbers again and again.
2. Iterative algorithm is storing computed values and building bottom up the final value. Memoization.

Dynamic Programming:

Finding a recursion that can be effectively/efficiently memoized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.

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13.1.2
Automatic/implicit memoization

Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (Fib(n) was previously computed)
        return stored value of Fib(n)
    else
        return Fib(n − 1) + Fib(n − 2)
```

How do we keep track of previously computed values?

Two methods: explicitly and implicitly (via data structure)

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How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)

Automatic memoization in python3...

```
#! /bin/python3
import functools
import time

@functools.cache
def binom_mem(n, i):
    if  ( i <= 0 ):
        return 1
    if  ( i >= n ):
        return 1
    return  binom_mem(n-1,i-1) + binom_mem(n-1,i)

def binom_reg(n, i):
    if  ( i <= 0 ):
        return 1
    if  ( i >= n ):
        return 1
    return  binom_reg(n-1,i-1) + binom_reg(n-1,i)

start = time.time()
print( binom_mem( 400, 200) )
end = time.time()
print( "Computing binom(400, 200) with  memozation: ", end - start)

start = time.time()
print( "binom(30, 15):", binom_reg( 30, 15) )
end = time.time()
print( "Computing binom(30, 15) with NO memozation: ", end - start)
```

Running it:

```
Computing binom(400, 200) with  memozation:  0.012813568115234375
binom(30, 15): 155117520
Computing binom(30, 15) with NO memozation:  20.24474811553955
```

Automatic implicit memoization

Initialize a (dynamic) dictionary data structure D to empty

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (n is already in D)
        return value stored with n in D
    val ⇐ Fib(n − 1) + Fib(n − 2)
    Store (n, val) in D
    return val
```

Use hash-table or a map to remember which values were already computed.

Explicit memoization (not automatic)

1. Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \ldots, n$.
2. Resulting code:

   $\textsf{Fib}(n)$:

   ```
   if (n = 0)
       return 0
   if (n = 1)
       return 1
   if (M[n] ≠ −1) // M[n]: stored value of Fib(n)
       return M[n]
   M[n] ⇐ Fib(n − 1) + Fib(n − 2)
   return M[n]
   ```
3. Need to know upfront the number of subproblems to allocate memory.

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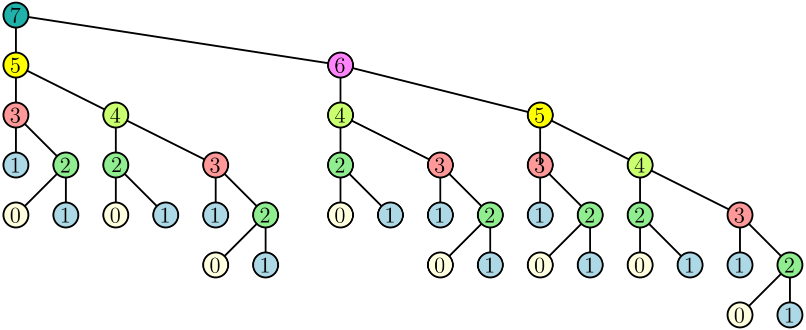
Recursion tree for the memoized Fib...



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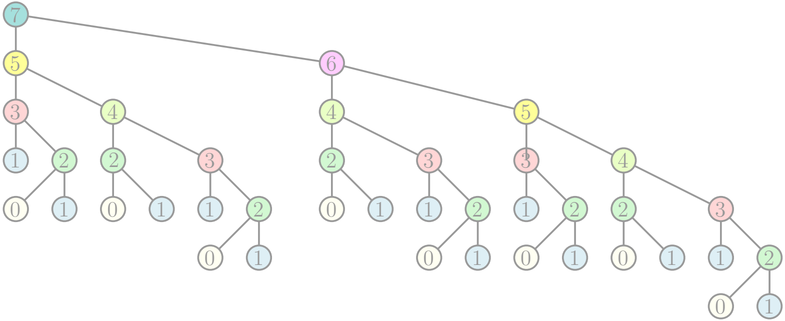

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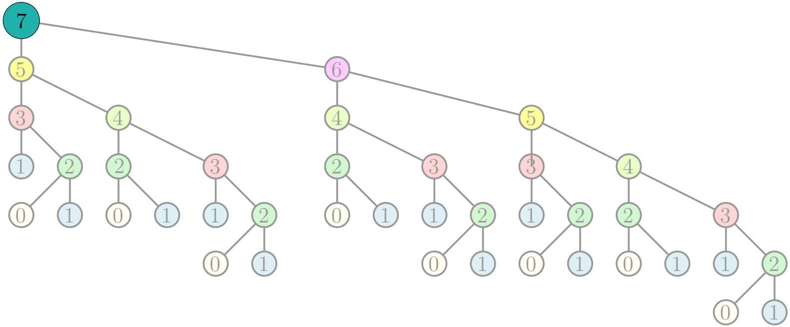

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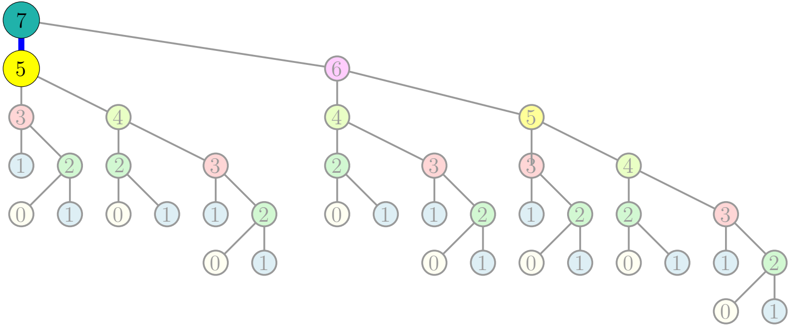

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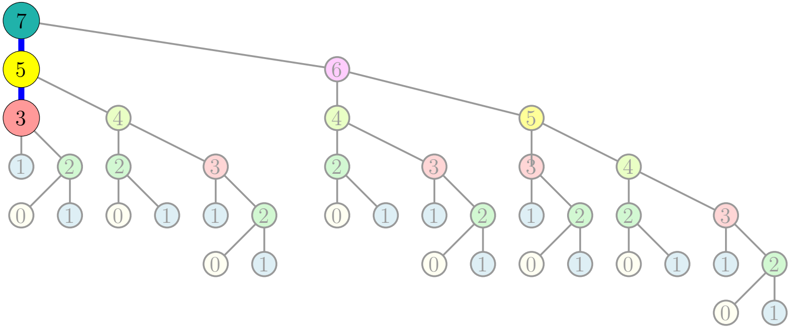

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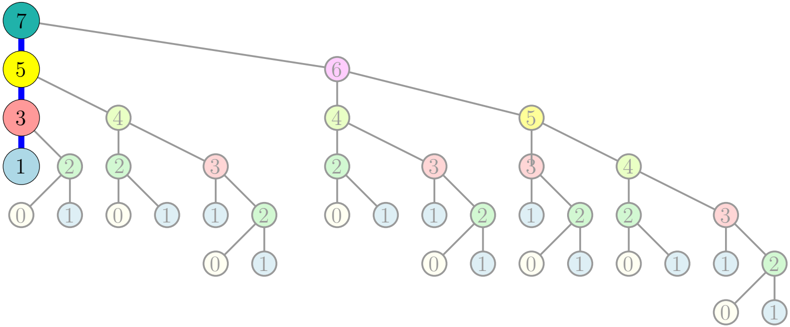

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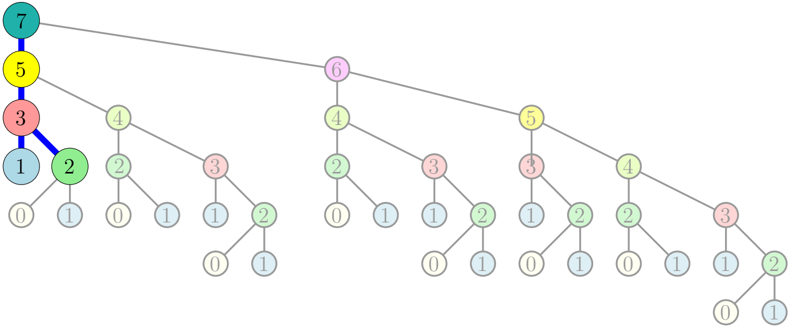

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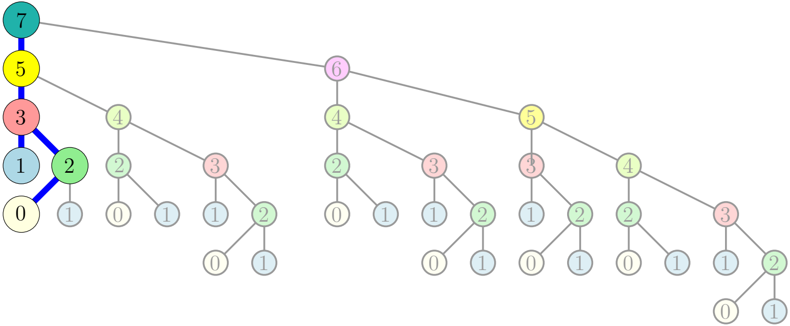

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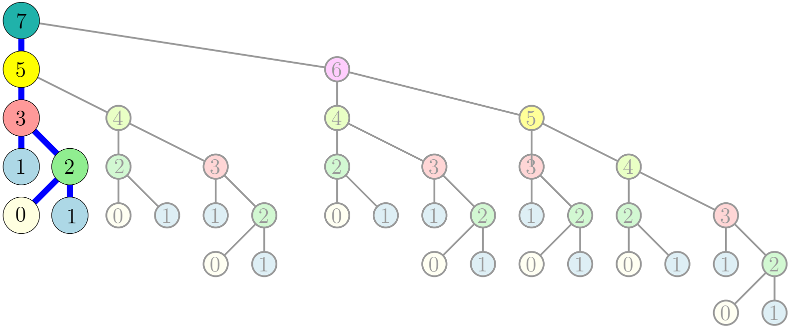

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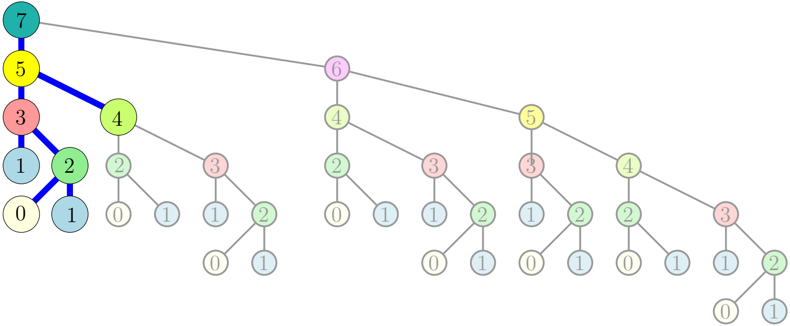

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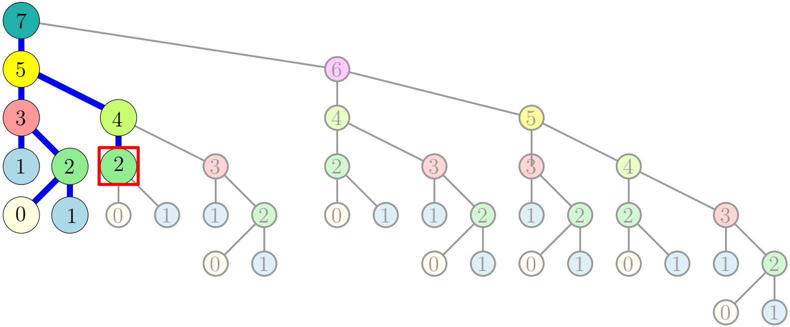

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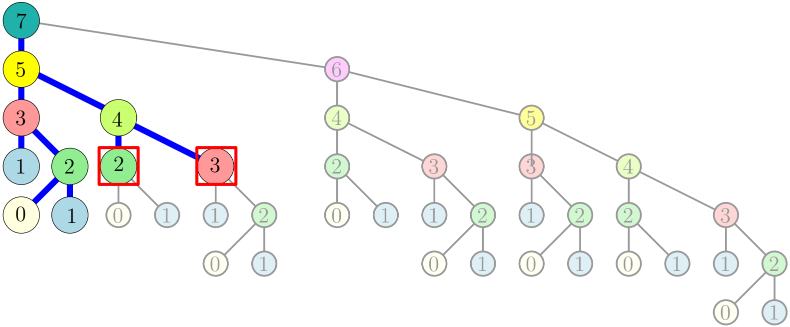

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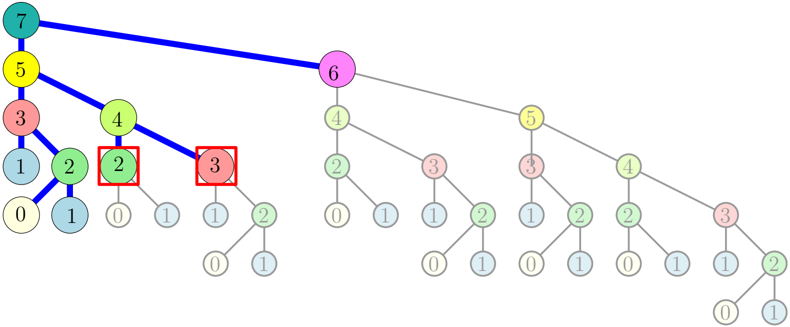

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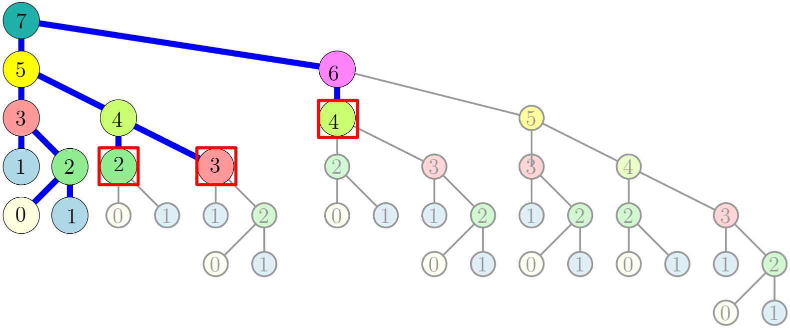

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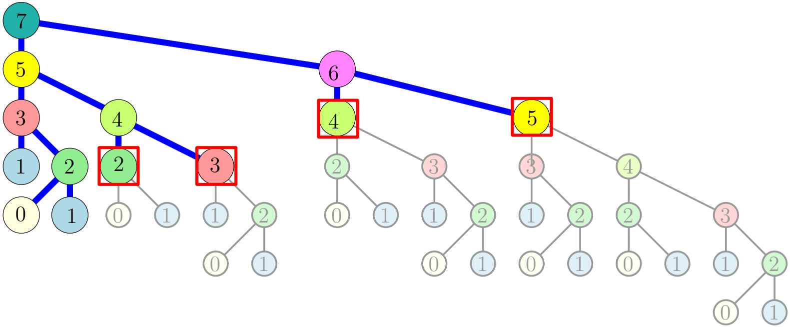

Automatic Memoization

1. Recursive version:

```
f(x_1, x_2, ..., x_d):
        CODE
```

2. Recursive version with memoization:

```
g(x_1, x_2, ..., x_d):
        if f already computed for (x_1, x_2, ..., x_d) then
            return value already computed
        NEW_CODE
```

3. NEW_CODE:
   3.1 Replaces any "**return** α" with
   3.2 Remember "$f(x_1, \ldots, x_d) = \alpha$"; **return** α.

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f(x_1, x_2, ..., x_d):
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$\boldsymbol{f}(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_d})$:
    CODE

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$\boldsymbol{g}(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_d})$:
    if $\boldsymbol{f}$ `already computed for` $(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_d})$ **then**
        return `value already computed`
    `NEW_CODE`

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   3.1 Replaces any "**return** α" with
   3.2 Remember "$\boldsymbol{f}(\boldsymbol{x_1}, \ldots, \boldsymbol{x_d}) = \alpha$"; **return** α.

Explicit vs Implicit Memoization

1. Explicit memoization (on the way to iterative algorithm) preferred:
   1.1 analyze problem ahead of time
   1.2 Allows for efficient memory allocation and access.
2. Implicit (automatic) memoization:
   2.1 problem structure or algorithm is not well understood.
   2.2 Need to pay overhead of data-structure.
   2.3 Functional languages (e.g., LISP) automatically do memoization, usually via hashing based dictionaries.

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Explicit/implicit memoization for Fibonacci

```
Init:   M[i] = −1,  i = 0, ..., n.

Fib(k):
    if (k = 0)
        return 0
    if (k = 1)
        return 1
    if (M[k] ≠ −1)
        return M[n]
    M[k] ⇐ Fib(k − 1) + Fib(k − 2)
    return M[k]
```

Explicit memoization

```
Init:   Init dictionary D

Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (n is already in D)
        return value stored with n in D
        val ⇐ Fib(n − 1) + Fib(n − 2)
    Store (n, val) in D
    return val
```

Implicit memoization

How many distinct calls?

```
binom(t, b)    // computes (t choose b)
    if t = 0 then return 0
    if b = t or b = 0 then return 1
    return binom(t − 1, b − 1) + binom(t − 1, b).
```

How many distinct calls does **binom**$(n, \lfloor n/2 \rfloor)$ makes during its recursive execution?

(A) $\Theta(1)$.

(B) $\Theta(n)$.

(C) $\Theta(n \log n)$.

(D) $\Theta(n^2)$.

(E) $\Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right)$.

That is, if the algorithm calls recursively **binom**$(17, 5)$ about 5000 times during the computation, we count this is a single distinct call.

Running time of memoized binom?

```
D: Initially an empty dictionary.
binomM(t, b)    // computes (t choose b)
    if b = t then return 1
    if b = 0 then return 0
    if D[t, b] is defined then return D[t, b]
    D[t, b] ⇐ binomM(t − 1, b − 1) + binomM(t − 1, b).
    return D[t, b]
```

Assuming that every arithmetic operation takes $O(1)$ time, What is the running time of $\text{binomM}(n, \lfloor n/2 \rfloor)$?

(A) $\Theta(1)$.

(B) $\Theta(n)$.

(C) $\Theta(n^2)$.

(D) $\Theta(n^3)$.

(E) $\Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right)$.

13.2
Dynamic programming

Removing the recursion by filling the table in the right order

"Dynamic programming"

```
Fib(n):
    if (n = 0)
        return 0
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    if (M[n] ≠ −1)
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```
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        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

Dynamic programming: Saving space!

Saving space. Do we need an array of n numbers? Not really.

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    prev2 = 0
    prev1 = 1
    for i = 2 to n do
        temp = prev1 + prev2
        prev2 = prev1
        prev1 = temp

    return prev1
```

Dynamic programming – quick review

Dynamic Programming is smart recursion

+ explicit memoization

+ filling the table in right order

+ removing recursion.

Dynamic programming – quick review

Dynamic Programming is smart recursion
+ explicit memoization
+ filling the table in right order
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Dynamic programming – quick review

Dynamic Programming is smart recursion
$+$ explicit memoization
$+$ filling the table in right order
$+$ removing recursion.

Analyzing memoized recursive function

Question: Suppose we have a recursive program $foo(x)$ that takes an input x.

- On input of size n the number of distinct sub-problems that $foo(x)$ generates is at most $A(n)$
- $foo(x)$ spends at most $B(n)$ time not counting the time for its recursive calls.

Suppose we memoize the recursion.

Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

Q: What is an upper bound on the running time of memoized version of $foo(x)$ if $|x| = n$? $O(A(n)B(n))$.

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13.2.1
Fibonacci numbers are big – corrected running time analysis

Back to Fibonacci Numbers

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    prev2 = 0
    prev1 = 1
    for i = 2 to n do
        temp = prev1 + prev2
        prev2 = prev1
        prev1 = temp

    return prev1
```

Is the iterative algorithm a polynomial time algorithm? Does it take $O(n)$ time?

1. input is n and hence input size is $\Theta(\log n)$
2. output is $F(n)$ and output size is $\Theta(n)$. Why?
3. Hence output size is exponential in input size so no polynomial time algorithm possible!
4. Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?

13.3
Checking if a string is in L^*

Problem

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsInL**$(string\ \ x)$ that decides whether x is in L

Goal Decide if $w \in L^*$ using **IsInL**$(string\ \ x)$ as a black box sub-routine

Problem

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsInL**($string\ x$) that decides whether x is in L

Goal Decide if $w \in L^{*}$ using **IsInL**($string\ x$) as a black box sub-routine

Problem

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsInL**$(\textit{string} \;\; x)$ that decides whether x is in L

$$\ast$$

Goal Decide if $w \in L$ using **IsInL**$(\textit{string} \;\; x)$ as a black box sub-routine

Problem

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsInL**$(string\ \ x)$ that decides whether x is in L

Goal Decide if $w \in L$ using **IsInL**$(string\ \ x)$ as a black box sub-routine

Problem

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsInL**(*string* x) that decides whether x is in L

$$*$$

Problem

Input: A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsInL**$(string\ \ x)$ that decides whether x is in L

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Problem

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsInL**$(string\ \ x)$ that decides whether x is in L

Goal Decide if using **IsInL**$(string\ \ x)$ as a black box sub-routine

Example 13.1.

Suppose L is *English* and we have a procedure to check whether a string/word is in the *English* dictionary.

- Is the string "isthisanenglishsentence" in $English^*$?
- Is "stampstamp" in $English^*$?
- Is "zibzzzad" in $English^*$?

Recursive Solution

When is $w \in L^*$?

$w \in L^* \iff w \in L$ or if $w = uv$ where $u \in L^*$ and $v \in L$, $|v| \geq 1$.

Assume w is stored in array $A[1..n]$

```
IsInL*(A[1..n]):
    If (n = 0) Output YES
    If (IsInL(A[1..n]))
        Output YES
    Else
        For (i = 1 to n − 1) do
            If IsInL*(A[1..i]) and IsInL(A[i + 1..n])
                Output YES

    Output NO
```

Recursive Solution

When is $w \in L^*$?

$w \in L^* \iff w \in L$ or if $w = uv$ where $u \in L^*$ and $v \in L$, $|v| \geq 1$.

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```

Question: How many distinct sub-problems does $\mathsf{IsInL}^*(A[1..n])$ generate? $O(n)$

Recursive Solution

Assume w is stored in array $A[1..n]$

```
IsInL*(A[1..n]):
    If (n = 0) Output YES
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            If IsInL*(A[1..i]) and IsInL(A[i + 1..n])
                Output YES

    Output NO
```

Question: How many distinct sub-problems does IsInL*$(A[1..n])$ generate? $O(n)$

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Assume w is stored in array $A[1..n]$

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Question: How many distinct sub-problems does $\mathsf{IsInL}^*(A[1..n])$ generate? $O(n)$

Example

Consider string ***samiam***

Naming subproblems and recursive equation

After seeing that number of subproblems is $O(n)$ we name them to help us understand the structure better.

$\textsf{ISL}^*(i)$: a boolean which is $\mathbf{1}$ if $A[1..i]$ is in L^*, $\mathbf{0}$ otherwise

Base case: $\textsf{ISL}^*(0) = \mathbf{1}$ interpreting $A[1..0]$ as ϵ

Recursive relation:

- $\textsf{ISL}^*(i) = 1$ if $\exists j, 0 \leq j < i$ s.t $\textsf{ISL}^*(j)$ and $\textsf{IsInL}(A[j+1..i])$
- $\textsf{ISL}^*(i) = 0$ otherwise

Output: $\textsf{ISL}^*(n)$

Naming subproblems and recursive equation

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Output: $\mathrm{ISL}^*(n)$

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After seeing that number of subproblems is $O(n)$ we name them to help us understand the structure better.

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Recursive relation:

- ISL*$(i) = 1$ if
  $\exists j$, $0 \leq j < i$ s.t ISL*(j) and IsInL$(A[j+1..i])$
- ISL*$(i) = 0$ otherwise

Output: ISL*(n)

Removing recursion to obtain iterative algorithm

Typically, after finding a dynamic programming recursion, we often convert the recursive algorithm into an iterative algorithm via explicit memoization and bottom up computation.

Why? Mainly for further optimization of running time and space.

How?

- First, allocate a data structure (usually an array or a multi-dimensional array that can hold values for each of the subproblems)
- Figure out a way to order the computation of the sub-problems starting from the base case.

Caveat: Dynamic programming is not about filling tables. It is about finding a smart recursion. First, find the correct recursion.

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Iterative Algorithm

```
IsStringinLstar-Iterative(A[1..n]):
    boolean ISL*[0..(n + 1)]
    ISL*[0] = TRUE
    for i = 1 to n do
        for j = 0 to i − 1 do
            if (ISL*[j] and IsInL(A[j + 1..i]))
                ISL*[i] = TRUE
                break

    if (ISL*[n] = 1) Output YES
    else Output NO
```

- **Running time:** $O(n^2)$ (assuming call to **IsInL** is $O(1)$ time)
- **Space:** $O(n)$

Iterative Algorithm

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- **Space:** $O(n)$

Example

Consider string ***samiam***

13.4
Longest Increasing Subsequence Revisited

13.4.1
Longest Increasing Subsequence

Sequences

Definition 13.1.

Sequence: an ordered list $a_1, a_2, \ldots, a_n$. **Length** of a sequence is number of elements in the list.

Definition 13.2.

$a_{i_1}, \ldots, a_{i_k}$ is a **subsequence** of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition 13.3.

A sequence is **increasing** if $a_1 < a_2 < \ldots < a_n$. It is **non-decreasing** if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly **decreasing** and **non-increasing**.

Sequences

Example...

Example 13.4.

1. Sequence: $\mathbf{6, 3, 5, 2, 7, 8, 1, 9}$
2. Subsequence of above sequence: $\mathbf{5, 2, 1}$
3. Increasing sequence: $\mathbf{3, 5, 9, 17, 54}$
4. Decreasing sequence: $\mathbf{34, 21, 7, 5, 1}$
5. Increasing <u>subsequence</u> of the first sequence: $\mathbf{2, 7, 9}$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an **increasing subsequence** $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example 13.5.

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an **increasing subsequence** $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example 13.5.

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8

Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

$\mathrm{LIS}(A[1..n])$:

1. Case 1: Does not contain $A[n]$ in which case $\mathrm{LIS}(A[1..n]) = \mathrm{LIS}(A[1..(n-1)])$
2. Case 2: contains $A[n]$ in which case $\mathrm{LIS}(A[1..n])$ is not so clear.

Observation 13.6.

For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is **LIS_smaller**$(A[1..n], x)$ *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

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Recursive Approach

$\boldsymbol{LIS(A[1..n])}$: the length of longest increasing subsequence in $\boldsymbol{A}$

LIS_smaller$\boldsymbol{(A[1..n], x)}$: length of longest increasing subsequence in $\boldsymbol{A[1..n]}$ with all numbers in subsequence less than $\boldsymbol{x}$

```
LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i − 1], x)
    if A[i] < x then
        m = max(m, 1 + LIS_smaller(A[1..i − 1], A[i]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

Recursive Approach

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LIS_smaller(A[1..i], x):
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LIS(A[1..n]):
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```

- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate? $O(n^2)$
- What is the running time if we memoize recursion? $O(n^2)$ since each call takes $O(1)$ time to assemble the answers from to recursive calls and no other computation.
- How much space for memoization? $O(n^2)$

Recursive Approach

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Naming subproblems and recursive equation

After seeing that number of subproblems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add ∞ at end of array (in position $n+1$)

$\mathrm{LIS}(i,j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

Base case: $\mathrm{LIS}(0,j) = 0$ for $1 \leq j \leq n+1$

Recursive relation:

- $\mathrm{LIS}(i,j) = \mathrm{LIS}(i-1,j)$ if $A[i] > A[j]$
- $\mathrm{LIS}(i,j) = \max\{LIS(i-1,j), 1 + LIS(i-1,i)\}$ if $A[i] \leq A[j]$

Output: $\mathrm{LIS}(n, n+1)$.

Assumption: $A[n+1] = +\infty$.

Naming subproblems and recursive equation

After seeing that number of subproblems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add ∞ at end of array (in position $n+1$)

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- $\mathrm{LIS}(i,j) = \mathrm{LIS}(i-1,j)$ if $A[i] > A[j]$
- $\mathrm{LIS}(i,j) = \max\{LIS(i-1,j), 1 + LIS(i-1,i)\}$ if $A[i] \le A[j]$

Output: $\mathrm{LIS}(n, n+1)$.
Assumption: $A[n+1] = +\infty$.

How to order bottom up computation?

$i \Downarrow$ \ j	1	2	3	4	5	6	7	$8 = n+1$
0								
1								
2								
3								
4								
5								
6								
7								

Recursive relation:

$$\mathrm{LIS}(i,j) = \begin{cases} 0 & i = 0 \\ \mathrm{LIS}(i-1,j) & A[i] > A[j] \\ \max \begin{cases} \mathrm{LIS}(i-1,j) \\ 1 + \mathrm{LIS}(i-1,i) \end{cases} & A[i] \le A[j] \end{cases}$$

Sequence: $A[1..7] = 6, 3, 5, 2, 7, 8, 1$ and $A[8] = +\infty$.

Iterative algorithm

The dynamic program for longest increasing subsequence

LIS-Iterative($A[1..n]$):
$\quad A[n+1] = \infty$
$\quad$ `int` $LIS[0..n, 1..n+1]$
$\quad$ **for** $j = 1 \dots n+1)$ **do** $LIS[0, j] = 0$

$\quad$ **for** $i = 1 \dots n)$ **do**
$\quad\quad$ **for** $(j = i+1 \dots n$ **do**
$\quad\quad\quad$ **if** $(A[i] > A[j])$
$\quad\quad\quad\quad LIS[i, j] = LIS[i-1, j]$
$\quad\quad\quad$ **else**
$\quad\quad\quad\quad LIS[i, j] = \max(LIS[i-1, j], 1 + LIS[i-1, i])$

$\quad$ `Return` $LIS[n, n+1]$

Running time: $O(n^2)$
Space: $O(n^2)$

Two comments

Question: Can we compute an optimum solution and not just its value?

Yes! See notes.

Question: Is there a faster algorithm for LIS? Yes! Using a different recursion and optimizing one can obtain an $O(n \log n)$ time and $O(n)$ space algorithm. $O(n \log n)$ time is not obvious. Depends on improving time by using data structures on top of dynamic programming.

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13.5: Dynamic programming via DAGs

andén

From wikipedia: An andén (plural andenes), Spanish for "platform", is a stair-step like terrace dug into the slope of a hillside for agricultural purposes. The term is most often used to refer to the terraces built by pre-Columbian cultures in the Andes mountains of South America. Andenes had several functions, the most important of which was to increase the amount of cultivatable land available to farmers by leveling a planting area for crops. The best known andenes are in Peru, especially in the Sacred Valley near the Inca capital of Cuzco and in the Colca Canyon. Many andenes have survived for more than 500 years and are still in use by farmers throughout the region.

Is it really dynamic programming?

Consider a sequence $\widehat{\alpha} \equiv \alpha_1, \ldots, \alpha_n$ of n distinct numbers, and a parameter $\delta > 0$. The sequence $\widehat{\alpha}$ is a **δ-andén**, if there exists an index i, such that:

1. For all t, we have $|\alpha_t - \alpha_{t+1}| \leq \delta$.
2. For all $t < i$, we have $\alpha_t < \alpha_{t+1}$.
3. For all $t \geq i$, we have $\alpha_t > \alpha_{t+1}$.

(I.e., a δ-andén is a hill where the difference between consecutive values is at most δ.)

The problem

The input is an array $A[1 \ldots n]$, and a parameter δ. Describe an algorithm that computes the length of the longest subsequence of A that forms a δ-andén. Your algorithm needs to output the number of elements in this subsequence (and not the subsequence itself).

13.6
How to come up with dynamic programming algorithm: summary

Dynamic Programming

1. Find a "smart" recursion for the problem in which the number of distinct subproblems is small; polynomial in the original problem size.
2. Estimate the number of subproblems, the time to evaluate each subproblem and the space needed to store the value.
3. This gives an upper bound on the total running time if we use automatic/explicit memoization.
4. Come up with an explicit memoization algorithm for the problem.
5. Eliminate recursion and find an iterative algorithm.
6. ...need to find the right way or order the subproblems evaluation. Th is leads to an a dynamic programming algorithm.
7. Optimize the resulting algorithm further
8. ...
9. Get rich!

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13.7
Supplemental: Some experiments with memoization

Edit distance: different memoizations

Input size n	Running time in seconds DP	Partial	Implicit memoization
$\mathbf{1,250}$	$\mathbf{0.01}$	$\mathbf{0.04}$	$\mathbf{0.20}$
$\mathbf{2,500}$	$\mathbf{0.04}$	$\mathbf{0.15}$	$\mathbf{0.84}$
$\mathbf{5,000}$	$\mathbf{0.18}$	$\mathbf{0.64}$	$\mathbf{3.73}$
$\mathbf{10,000}$	$\mathbf{0.72}$	$\mathbf{2.50}$	$\mathbf{15.05}$
$\mathbf{20,000}$	$\mathbf{2.88}$	$\mathbf{9.91}$	$\mathbf{55.35}$
$\mathbf{40,000}$	$\mathbf{12.00}$	$\mathbf{40.00}$	out of memory

For the input n, two random strings of length n were generated, and their distance computed using edit distance.

Note, that edit-distance is simple enough to that DP gets very good performance. For more complicated problems, the advantage of DP would probably be much smaller.

The asymptotic running time here is $\Theta(n^2)$.

Edit distance: different memoizations

More details

1. The implementation was done in C++, using -O9 in compilation.
2. DP = Dynamic Programming = iterative implementation using arrays.
3. Partial memoization = Still uses recursive code, but remembers the results in tables that are managed directly by the code.
4. Implicit memoization = implemented using the standard `unordered_map`.

Edit distance: different memoizations

Conclusions

1. If you are in interview setup, you should probably solve the problem using DP. That what you would be expected to do.
2. Otherwise, I would probably implement partial memoization – it still has the simplicity of the recursive solution, while having a decent performance. If I really care about performance I would implement the DP.
3. Using implicit memoization probably makes sense only if running time is not really an issue.

13.8
Tangential: Fibonacci and his numbers

Fibonacci = Leonardo Bonacci

1. CE 1170–1250.
2. Italian. Spent time in Bugia, Algeria with his father (trader).
3. Traveled around the Mediterranean coast, learned the Hindu–Arabic numerals
4. Hindu–Arabic numerals:
   4.1 Developed before 400 CE by Hindu philosophers.
   4.2 Arrived to the Arab world sometime before 825CE.
   4.3 Muhammad ibn Musa al-Khwarizmi (Algorithm/Algebra) wrote a book in 825 CE explaining the new system. (Showed how to solved quadratic equations.)
5. 1202 CE: Fibonacci wrote a book "Liber Abaci" (book of calculations) that popularized the new system.
6. Brought and popularized the Hindu–Arabic system to Italy.

Fibonacci numbers

1. Fibonacci in Liber Abaci posed and solved a problem involving the growth of a population of rabbits based on idealized assumptions.
2. Describe growth processes.
   Every month a mature pair of rabbits give birth to one pair of young rabbits.

Month	grownup pairs	Young pairs
1	1	0
2	1	1
3	2	1
4	3	2
5	5	3
⋮	⋮	⋮
40	102,334,155	63,245,986

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Fibonacci numbers II

1. $\lim_{n\to\infty} F_n/F_{n-1} = \varphi$.
2. Golden ratio: $\varphi = (\sqrt{5}+1)/2 \approx 1.618033$.
3. For $a > b > 0$, $\varphi = \frac{a+b}{a} = \frac{a}{b}$. $\implies$ $\frac{\varphi+1}{\varphi} = \varphi$. $\implies$ $0 = \varphi^2 - \varphi - 1$.
4. $\varphi = \frac{1\pm\sqrt{1+4}}{2}$ since φ is not negative, so...
5. $F_n = \frac{\varphi^n-(1-\varphi)^n}{\sqrt{5}}$
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7. Many applications of GR and Fibonacci numbers in nature, models (stock market), art, etc...

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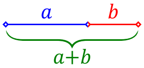

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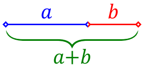

$a+b$ is to a as a is to b

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Fibonacci numbers: Binet's formula

1. $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi$ are solution to the equation: $x^2 = x + 1$.
2. As such, φ and ψ a solution to the equation: $x^n = x^{n-1} + x^{n-2}$.
3. Consider the sequence $U_n = U_{n-1} + U_{n-2}$.
   For any $\alpha, \beta \in \mathbb{R}$, consider $U_n = \alpha\varphi^n + \beta\psi^n$. A valid solution to the sequence.

$$U_n = U_{n-1} + U_{n-2} = \alpha\varphi^{n-1} + \beta\psi^{n-1} + \alpha\varphi^{n-2} + \beta\psi^{n-2}$$

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\begin{aligned}
U_n = U_{n-1} + U_{n-2} &= \alpha\varphi^{n-1} + \beta\psi^{n-1} + \alpha\varphi^{n-2} + \beta\psi^{n-2} \\
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\end{aligned}
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$$\begin{aligned} U_n = U_{n-1} + U_{n-2} &= \alpha\varphi^{n-1} + \beta\psi^{n-1} + \alpha\varphi^{n-2} + \beta\psi^{n-2} \\ &= \left(\alpha\varphi^{n-1} + \alpha\varphi^{n-2}\right) + \left(\beta\psi^{n-1} + \beta\psi^{n-2}\right) = U_{n-1} + U_{n-2}. \end{aligned}$$

4. Solve the system
   $U_0 = 0$ and $U_1 = 1 \iff \alpha\varphi^0 + \beta\psi^0 = 0$ and $\alpha\varphi^1 + \beta\psi^1 = 1$

Fibonacci numbers: Binet's formula

1. $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi$ are solution to the equation: $x^2 = x + 1$.
2. As such, φ and ψ a solution to the equation: $x^n = x^{n-1} + x^{n-2}$.
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4. Solve the system
   $U_0 = 0$ and $U_1 = 1 \iff \alpha\varphi^0 + \beta\psi^0 = 0$ and $\alpha\varphi^1 + \beta\psi^1 = 1 \implies \beta = -\alpha$

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   $\implies \alpha = 1/\sqrt{5} \implies F_n = U_n = \alpha\varphi^n + \beta\psi^n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$

Fibonacci numbers really fast

$$\begin{pmatrix} y \\ x+y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

As such,

$$\begin{aligned} \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n-2} \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} F_{n-3} \\ F_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n-3} \begin{pmatrix} F_2 \\ F_1 \end{pmatrix}. \end{aligned}$$

More on fast Fibonacci numbers

Continued

Thus, computing the nth Fibonacci number can be done by computing $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n-3}$.

Which can be done in $O(\log n)$ time (how?). What is wrong?