Halting, Undecidability, and Maybe Some Complexity

Lecture 9

Tuesday, September 24, 2024

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Quote

"Young man, in mathematics you don't understand things. You just get used to them."
– John von Neumann.

9.1
Cantor's diagonalization argument

You can not count the real numbers

$I = (0,1)$.
$\mathbb{N} = \{1,2,3,\ldots\}$ the integer numbers

Claim 9.1 (Cantor).

$|\mathbb{N}| \neq |I|$

Claim 9.2 (Warm-up).

$|\mathbb{N}| \leq |I|$

Proof.

$|\mathbb{N}| \leq |I|$ exists a one-to-one mapping from $\mathbb{N}$ to I. One such mapping is $f(i) = 1/i$, which readily implies the claim. □

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You can not count the real numbers II

$I = (0, 1)$, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Claim 9.3 (Cantor).

$|\mathbb{N}| \neq |I|$, *where* $I = (0, 1)$.

Proof.

Write every number in $(0, 1)$ in its decimal expansion. E.g.,
$1/3 = 0.33333333333333333333\ldots$.
Assume that $|\mathbb{N}| = |I|$. Then there exists a one-to-one mapping $f : \mathbb{N} \to I$. Let β_i be the ith digit of $f(i) \in (0, 1)$.
$d_i =$ any number in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \setminus \{d_{i-1}, \beta_i\}$
$D = 0.d_1 d_2 d_3 \ldots \in (0, 1)$.
D is a well defined unique number in $(0, 1)$,
But there is no j such that $f(j) = D$. A contradiction. $\square$

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The matrix...

	$f(1)$	$f(2)$	$f(3)$	$f(4)$	$\dots$
1	**1**	1	0	0	$\dots$
2	0	**1**	0	1	$\dots$
3	1	0	**1**	1	$\dots$
4	0	1	0	**0**	$\dots$
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$\implies \forall i \ \beta_i \neq d_i$.

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$D = 0.23232323\ldots$.
D can not be the i column, because $\beta_i \neq d_i$.

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D can not be the i column, because $\beta_i \neq d_i$.
But D can not be in the matrix...

The liar paradox

When one day an expedition was sent to the spatial coordinates that Voojagig had claimed for this planet they discovered only a small asteroid inhabited by a solitary old man who claimed repeatedly that nothing was true, though he was later discovered to be lying.
– The Hitchhiker Guide to the Galaxy

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2. Related to Russell's paradox.
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9.2
Introduction to the halting theorem

The halting problem

Halting problem: Given a program Q, if we run it would it stop?

Q: Can one build a program P, that always stops, and solves the halting problem.

Theorem 9.1 ("Halting theorem").

There is no program that always stops and solves the halting problem.

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Intuition, why solving the Halting problem is really hard

Definition 9.2.

An integer number n is a **weird number** *if*

- *the sum of the proper divisors (including 1 but not itself) of n the number is $> n$,*
- *no subset of those divisors sums to the number itself.*

70 is weird. Its divisors are $1, 2, 5, 7, 10, 14, 35$.
$1 + 2 + 5 + 7 + 10 + 14 + 35 = 74$. No subset of them adds up to 70.

Open question: Are there are any odd weird numbers?

Write a program P that tries all odd numbers in order, and check if they are weird. The programs stops if it found such number.

If can solve halting problem $\implies$ can resolve this open problem.

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If you can halt, you can prove or disprove anything...

1. Consider any math claim C.
2. Prover algorithm P_C:
   (A) Generate sequence of all possible proofs (sequence of strings) into a pipe/queue.
   (B) $\langle p \rangle \leftarrow$ pop top of queue.
   (C) Feed $\langle p \rangle$ and $\langle C \rangle$, into a proof verifier ("easy").
   (D) If $\langle p \rangle$ valid proof of $\langle C \rangle$, then stop and accept.
   (E) Go to (B).
3. P_C halts $\iff$ C is true and has a proof.
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9.3
The halting theorem

Encodings

M: Turing machine
$\langle M \rangle$: a binary string uniquely describing M (i.e., it is a number.
w: An input string.
$\langle M, w \rangle$: A unique binary string encoding both M and input w.

$$\mathrm{A}_{\mathrm{TM}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } \mathrm{TM} \text{ and } M \text{ accepts } w \right\}.$$

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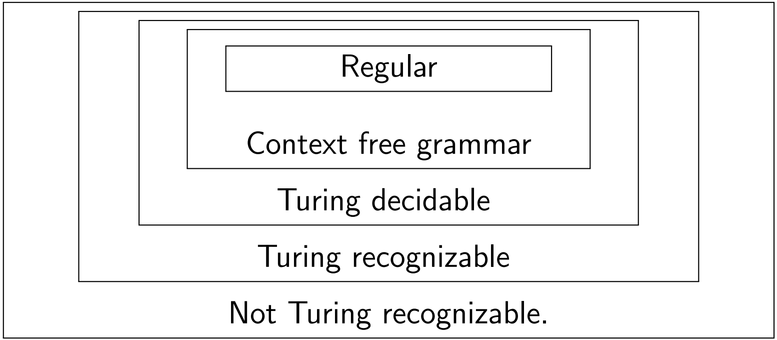
Complexity classes



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Lemma 9.1.

$\mathbf{A}_{\mathrm{TM}}$ *is Turing recognizable.*

Proof.

Input: $\langle M, w \rangle$.

Using UTM simulate running M on w. If M accepts w then accept, if M rejects then reject. Otherwise, the simulation runs forever. □

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Theorem 9.2 (The halting theorem.).

$\mathbf{A}_{\mathrm{TM}}$ *is not Turing decidable.*

Proof: Assume $\mathbf{A}_{\mathrm{TM}}$ is TM decidable...

Halt: TM deciding $\mathbf{A}_{\mathrm{TM}}$. **Halt** always halts, and works as follows:

$$\mathbf{Halt}\Big(\langle M, w\rangle\Big) = \begin{cases} \text{accept} & M \text{ accepts } w \\ \text{reject} & M \text{ does not accept } w. \end{cases}$$

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Halting theorem proof continued 1

We build the following new function:

```
Flipper( ⟨M⟩)
  res ← Halt(⟨M, M⟩)
  if res is accept then
          reject
  else
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```

Flipper always stops:

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Flipper is a TM (duh!), and as such it has an encoding ⟨**Flipper**⟩. Run **Flipper** on itself:

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This is absurd. Ridiculous even!

Assumption that **Halt** exists is false. $\implies$ $\mathbf{A}_{\mathrm{TM}}$ is not TM decidable. □

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But where is the diagonalization argument????

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\dots$
M_1	**rej**	acc	rej	rej	$\dots$
M_2	rej	**acc**	rej	acc	$\dots$
M_3	acc	acc	**acc**	rej	$\dots$
M_4	rej	acc	acc	**rej**	$\dots$
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9.4
Unrecognizable

TM recognizable

Definition 9.1.

Language L is TM **decidable** *if there exists M that always stops, such that $L(M) = L$.*

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Language L is TM **recognizable** *if there exists M that stops on some inputs, such that $L(M) = L$.*

Theorem 9.3 (Halting).

$A_{\mathrm{TM}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a TM and } M \text{ accepts } w \right\}$. *is TM recognizable, but not decidable.*

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Lemma 9.4.

If L and $\overline{L} = \Sigma^* \setminus L$ are both TM recognizable, then L and $\overline{L}$ are decidable.

Proof.

M: TM recognizing L.
M_c: TM recognizing $\overline{L}$.
Given input x, using UTM simulating running M and M_c on x in parallel. One of them must stop and accept. Return result.
$\implies L$ is decidable. □

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Complement language for $\mathbf{A}_{\mathsf{TM}}$

$$\overline{\mathbf{A}_{\mathrm{TM}}} = \Sigma^* \setminus \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } \mathrm{TM} \text{ and } M \text{ accepts } w \right\}.$$

But don't really care about invalid inputs. So, really:

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Theorem 9.5.

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Proof.

$\mathbf{A}_{\mathrm{TM}}$ is TM-recognizable.
If $\overline{\mathbf{A}_{\mathrm{TM}}}$ is TM-recognizable
$\Longrightarrow$ (by Lemma)
$\mathbf{A}_{\mathrm{TM}}$ is decidable. A contradiction. □

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Theorem 9.5.

The language

$$\overline{\mathbf{A}_{\mathrm{TM}}} = \left\{ \langle M, w \rangle \;\middle|\; M \textit{ is a } \mathrm{TM} \textit{ and } M \textit{ does } \textbf{not} \textit{ accept } w \right\}.$$

is not TM *recognizable.*

Proof.

$\mathbf{A}_{\mathrm{TM}}$ is TM-recognizable.
If $\overline{\mathbf{A}_{\mathrm{TM}}}$ is TM-recognizable
$\Longrightarrow$ (by Lemma)
$\mathbf{A}_{\mathrm{TM}}$ is decidable. A contradiction. $\square$

9.5
Turing complete

Equivalent to a program

Definition 9.1.

A system is **Turing complete** *if one can simulate a Turing machine using it.*

1. Programming languages (yey!).
2. C++ templates system (boo).
3. John Conway's game of life.
4. Many games (Minesweeper).
5. Post's correspondence problem.

Equivalent to a program

Definition 9.1.

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5. Post's correspondence problem.

Post's correspondence problem

$\boldsymbol{S}$: set of **domino tiles**.

$\boxed{\begin{array}{c} \boldsymbol{abb} \\ \hline \boldsymbol{bc} \end{array}}$: domino piece a string at the top and a string at the bottom.

Example:

$$S = \left\{ \boxed{\begin{array}{c} b \\ \hline ca \end{array}}, \boxed{\begin{array}{c} a \\ \hline ab \end{array}}, \boxed{\begin{array}{c} ca \\ \hline a \end{array}}, \boxed{\begin{array}{c} abc \\ \hline c \end{array}} \right\}.$$

Matching dominos

$$S = \left\{ \begin{bmatrix} b \\ \hline ca \end{bmatrix}, \begin{bmatrix} a \\ \hline ab \end{bmatrix}, \begin{bmatrix} ca \\ \hline a \end{bmatrix}, \begin{bmatrix} abc \\ \hline c \end{bmatrix} \right\}.$$

match for S: ordered list of dominos from S, such that top strings make same string as bottom strings. Example:

$$\begin{bmatrix} a \\ \hline ab \end{bmatrix} \begin{bmatrix} b \\ \hline ca \end{bmatrix} \begin{bmatrix} ca \\ \hline a \end{bmatrix} \begin{bmatrix} a \\ \hline ab \end{bmatrix} \begin{bmatrix} abc \\ \hline c \end{bmatrix}.$$

(1) Can use same domino more than once.
(2) Do not have to use all pieces of S.

Matching dominos

$$S = \left\{ \boxed{\dfrac{b}{ca}}, \boxed{\dfrac{a}{ab}}, \boxed{\dfrac{ca}{a}}, \boxed{\dfrac{abc}{c}} \right\}.$$

match for S: ordered list of dominos from S, such that top strings make same string as bottom strings. Example:

$$\boxed{\dfrac{a}{ab}}\ \boxed{\dfrac{b}{ca}}\ \boxed{\dfrac{ca}{a}}\ \boxed{\dfrac{a}{ab}}\ \boxed{\dfrac{abc}{c}}.$$

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$$\boxed{\frac{a}{ab}} \boxed{\frac{b}{ca}} \boxed{\frac{ca}{a}} \boxed{\frac{a}{ab}} \boxed{\frac{abc}{c}}.$$

(1) Can use same domino more than once.
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Post's Correspondence Problem

Post's Correspondence Problem (PCP) is deciding whether a set of dominos has a match or not.
modified Post's Correspondence Problem (MPCP): PCP + a special tile.
Matches for MPCP have to start with the special tile.

Theorem 9.2.

The MPCP *problem is undecidable.*