Context Free Languages and Grammars

Lecture 7

Tuesday, September 17, 2024

LaTeXed: September 17, 2024 10:24

7.1
Outputting a random balanced strings

Outputting a random balanced string

```
function  S()
    r = rand(1:5)
    if  r == 1
        S()
        S()
    elseif  r ∈ 2:4
        print("(")
        S()
        print(")")
    end
end

S()
println( "\n" )
```

7.2
A fluffy introduction to context free languages, push down automatas

What stack got to do with it?

What's a stack but a second hand memory?

1. DFA/NFA/Regular expressions.
   $\equiv$ constant memory computation.
2. NFA $+$ stack
   $\equiv$ context free grammars (CFG).
3. Turing machines DFA/NFA $+$ unbounded memory.
   $\equiv$ a standard computer/program.
   $\equiv$ NFA with two stacks.

Context Free Languages and Grammars

- Programming Language Specification
- Parsing
- Natural language understanding
- Generative model giving structure
- ...

Programming Languages

```
<relational-expression> ::= <shift-expression>
                          | <relational-expression> < <shift-expression>
                          | <relational-expression> > <shift-expression>
                          | <relational-expression> <= <shift-expression>
                          | <relational-expression> >= <shift-expression>

<shift-expression> ::= <additive-expression>
                     | <shift-expression> << <additive-expression>
                     | <shift-expression> >> <additive-expression>

<additive-expression> ::= <multiplicative-expression>
                        | <additive-expression> + <multiplicative-expression>
                        | <additive-expression> - <multiplicative-expression>

<multiplicative-expression> ::= <cast-expression>
                              | <multiplicative-expression> * <cast-expression>
                              | <multiplicative-expression> / <cast-expression>
                              | <multiplicative-expression> % <cast-expression>

<cast-expression> ::= <unary-expression>
                    | ( <type-name> ) <cast-expression>

<unary-expression> ::= <postfix-expression>
                     | ++ <unary-expression>
                     | -- <unary-expression>
                     | <unary-operator> <cast-expression>
                     | sizeof <unary-expression>
                     | sizeof <type-name>

<postfix-expression> ::= <primary-expression>
                       | <postfix-expression> [ <expression> ]
                       | <postfix-expression> ( {<assignment-expression>}* )
                       | <postfix-expression> . <identifier>
                       | <postfix-expression> -> <identifier>
                       | <postfix-expression> ++
                       | <postfix-expression> --
```

Natural Language Processing

English sentences can be described as

$$
\begin{aligned}
&\langle S\rangle \rightarrow \langle NP\rangle\langle VP\rangle\\
&\langle NP\rangle \rightarrow \langle CN\rangle \mid \langle CN\rangle\langle PP\rangle\\
&\langle VP\rangle \rightarrow \langle CV\rangle \mid \langle CV\rangle\langle PP\rangle\\
&\langle PP\rangle \rightarrow \langle P\rangle\langle CN\rangle\\
&\langle CN\rangle \rightarrow \langle A\rangle\langle N\rangle\\
&\langle CV\rangle \rightarrow \langle V\rangle \mid \langle V\rangle\langle NP\rangle\\
&\langle A\rangle \rightarrow \text{a} \mid \text{the}\\
&\langle N\rangle \rightarrow \text{boy} \mid \text{girl} \mid \text{flower}\\
&\langle V\rangle \rightarrow \text{touches} \mid \text{likes} \mid \text{sees}\\
&\langle P\rangle \rightarrow \text{with}
\end{aligned}
$$

English Sentences

Examples

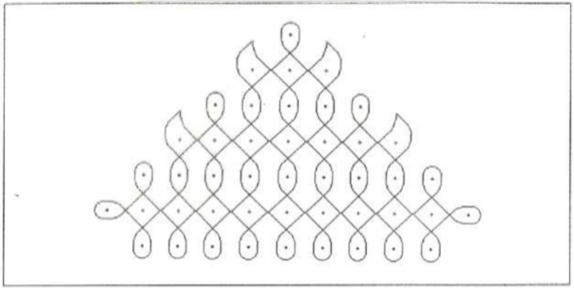
$$\overbrace{\underbrace{\text{a}}_{\text{article}}\ \underbrace{\text{boy}}_{\text{noun}}}^{\text{noun-phrs}}\ \overbrace{\underbrace{\text{sees}}_{\text{verb}}}^{\text{verb-phrs}}$$

$$\overbrace{\underbrace{\text{the}}_{\text{article}}\ \underbrace{\text{boy}}_{\text{noun}}}^{\text{noun-phrs}}\ \overbrace{\underbrace{\text{sees}}_{\text{verb}}\ \underbrace{\text{a flower}}_{\text{noun-phrs}}}^{\text{verb-phrs}}$$

Models of Growth

- *L*-systems
- http://www.kevs3d.co.uk/dev/lsystems/

Kolam drawing generated by grammar



7.3
Formal definition of convex-free languages (CFGs)

Context Free Grammar (CFG) Definition

Definition 7.1.

A CFG is a quadruple $G = (V, T, P, S)$

- V is a finite set of non-terminal symbols
- T is a finite set of terminal symbols (alphabet)
- P is a finite set of productions, each of the form
  $A \rightarrow \alpha$
  where $A \in V$ and α is a string in $(V \cup T)^*$.
  Formally, $P \subset V \times (V \cup T)^*$.
- $S \in V$ is a start symbol

$$G = \big(\ \text{Variables},\quad \text{Terminals},\quad \text{Productions},\quad \text{Start var}\ \big)$$

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Example

- $V = \{S\}$
- $T = \{a, b\}$
- $P = \{S \to \epsilon \mid a \mid b \mid aSa \mid bSb\}$
  (abbrev. for $S \to \epsilon, S \to a, S \to b, S \to aSa, S \to bSb$)

$$S \rightsquigarrow aSa \rightsquigarrow abSba \rightsquigarrow abbSbba \rightsquigarrow abb\,b\,bba$$

What strings can S generate like this?

Example

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Example formally...

- $V = \{S\}$
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- $P = \{S \to \epsilon \mid a \mid b \mid aSa \mid bSb\}$
  (abbrev. for $S \to \epsilon, S \to a, S \to b, S \to aSa, S \to bSb$)

$$G = \left(\{S\}, \quad \{a, b\}, \quad \left\{ \begin{array}{c} S \to \epsilon, \\ S \to a, \\ S \to b \\ S \to aSa \\ S \to bSb \end{array} \right\} \quad S \right)$$

Palindromes

- Madam in Eden I'm Adam
- Dog doo? Good God!
- Dogma: I am God.
- A man, a plan, a canal, Panama
- Are we not drawn onward, we few, drawn onward to new era?
- Doc, note: I dissent. A fast never prevents a fatness. I diet on cod.
- http://www.palindromelist.net

Examples

$L = \{0^n 1^n \mid n \geq 0\}$

$S \to \epsilon \mid 0S1$

Examples

$L = \{0^n1^n \mid n \geq 0\}$

$S \rightarrow \epsilon \mid 0S1$

Notation and Convention

Let $G = (V, T, P, S)$ then

- $a, b, c, d, \ldots,$ in T (terminals)
- $A, B, C, D, \ldots,$ in V (non-terminals)
- $u, v, w, x, y, \ldots$ in T^* for strings of terminals
- $\alpha, \beta, \gamma, \ldots$ in $(V \cup T)^*$
- X, Y, X in $V \cup T$

"Derives" relation

Formalism for how strings are derived/generated

Definition 7.2 (derive).

Let $G = (V, T, P, S)$ be a CFG. For strings $\alpha_1, \alpha_2 \in (V \cup T)^*$: α_1 derives α_2 denoted by $\alpha_1 \rightsquigarrow_G \alpha_2$ if there exist strings β, γ, δ in $(V \cup T)^*$ such that

- $\alpha_1 = \beta A \delta$
- $\alpha_2 = \beta \gamma \delta$
- $A \rightarrow \gamma$ is in P.

Example 7.3.

For $S \rightarrow \epsilon \mid 0S1$

$S \rightsquigarrow \epsilon$, $S \rightsquigarrow 0S1$, $0S1 \rightsquigarrow 00S11$, $0S1 \rightsquigarrow 01$.

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"Derives" relation continued

Definition 7.4.

For integer $k \geq 0$, $\alpha_1 \rightsquigarrow^k \alpha_2$ inductive defined:

- $\alpha_1 \rightsquigarrow^0 \alpha_2$ if $\alpha_1 = \alpha_2$
- $\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow \beta_1$ and $\beta_1 \rightsquigarrow^{k-1} \alpha_2$.
- Alternative definition: $\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow^{k-1} \beta_1$ and $\beta_1 \rightsquigarrow \alpha_2$

$\rightsquigarrow^*$ is the reflexive and transitive closure of $\rightsquigarrow$.

$\alpha_1 \rightsquigarrow^* \alpha_2$ if $\alpha_1 \rightsquigarrow^k \alpha_2$ for some k.

Example 7.5.

For $S \rightarrow \epsilon \mid 0S1$

$\implies S \rightsquigarrow^* \epsilon$, $0S1 \rightsquigarrow^* 0000011111$.

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"Derives" relation continued

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- $\alpha_1 \leadsto^0 \alpha_2$ if $\alpha_1 = \alpha_2$
- $\alpha_1 \leadsto^k \alpha_2$ if $\alpha_1 \leadsto \beta_1$ and $\beta_1 \leadsto^{k-1} \alpha_2$.
- Alternative definition: $\alpha_1 \leadsto^k \alpha_2$ if $\alpha_1 \leadsto^{k-1} \beta_1$ and $\beta_1 \leadsto \alpha_2$

$\leadsto^*$ is the reflexive and transitive closure of $\leadsto$.

$\alpha_1 \leadsto^* \alpha_2$ if $\alpha_1 \leadsto^k \alpha_2$ for some k.

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For $S \to \epsilon \mid 0S1$
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Context Free Languages

Definition 7.6.

The language generated by CFG $G = (V, T, P, S)$ is denoted by $L(G)$ where $L(G) = \{w \in T^* \mid S \leadsto^* w\}$.

Definition 7.7.

A language L is context free (CFL) if it is generated by a context free grammar. That is, there is a CFG G such that $L = L(G)$.

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Example

$$L = \{0^n 1^n \mid n \geq 0\}$$

$$S \to \epsilon \mid 0S1$$

$$L = \{0^n 1^m \mid m > n\}$$

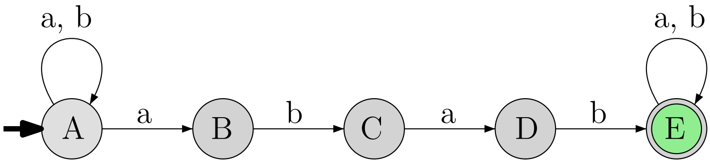
$$L = \left\{ w \in \{(,)\}^* \;\middle|\; w \text{ is properly nested string of parenthesis} \right\}.$$

7.4
Converting regular languages into CFL

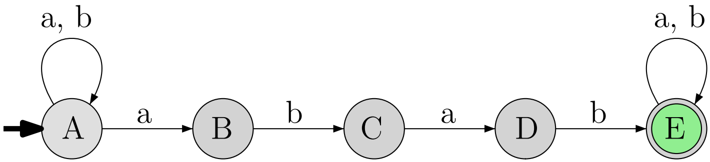
Converting regular languages into CFL

$M = (Q, \Sigma, \delta, s, A)$: DFA for regular language L.

$$G = \Big(\overbrace{Q}^{\text{Variables}}, \overbrace{\Sigma}^{\text{Terminals}}, \overbrace{\{q \to a\delta(q,a) \mid q \in Q, a \in \Sigma\} \cup \{q \to \varepsilon \mid q \in A\}}^{\text{Productions}}, \overbrace{s}^{\text{Start var}} \Big)$$



Conversion continued...



$$G = \left(\{A, B, C, D, E\}, \{a, b\}, \left\{ \begin{array}{c} A \rightarrow aA, A \rightarrow bA, A \rightarrow aB, \\ B \rightarrow bC, \\ C \rightarrow aD, \\ D \rightarrow bE, \\ E \rightarrow aE, E \rightarrow bE, E \rightarrow \varepsilon \end{array} \right\}, A \right)$$

The result...

Lemma 7.1.

For an regular language $\mathbf{L}$, there is a context-free grammar (CFG) *that generates it.*

7.5
CFL as a python program

0^n1^n

The grammar G:

$$S \to \varepsilon \mid 0S1$$

Can be translated into the python program:

```
#! /bin/python3
import random

# S → epsilon | 0 S 1
def S():
    match random.randrange(10):
        case 0:
            return   # epsilon
        case _:
            print( "0", end='' )
            S()
            print( "1", end='' )

S()
print( "" )
```

$L(G) =$ any string that this program might output.

Balanced parenthesis expression

The grammar G:

$$S \rightarrow \varepsilon \mid (S) \mid SS$$

Can be translated into the python program:

```
#! /bin/python3
import random

# S → epsilon  |  ( S )  |  S S
def  S():
    match  random.randrange(3):
        case 0:       # epsilon
            return

        case 1:       # ( S )
            print( "(", end='' )
            S()
            print( ")", end='' )

        case _:       # SS
            S()
            S()

S()
print( "" )
```

$L(G) =$ any string that this program might output.

7.6
Some properties of CFLs

7.6.1
Closure properties of CFLs

Bad news: Canonical non-CFL

Theorem 7.1.

$L = \{a^n b^n c^n \mid n \geq 0\}$ *is not context-free.*

Proof based on pumping lemma for CFLs. See supplemental for the proof.

More bad news: CFL not closed under intersection

Theorem 7.2.

CFLs *are* *not* *closed under intersection.*

Closure Properties of CFLs

$G_1 = (V_1, T, P_1, S_1)$ and $G_2 = (V_2, T, P_2, S_2)$
Assumption: $V_1 \cap V_2 = \emptyset$, that is, non-terminals are not shared

Theorem 7.3.
CFLs *are closed under union.* L_1, L_2 CFLs *implies* $L_1 \cup L_2$ *is a* CFL.

Theorem 7.4.
CFLs *are closed under concatenation.* L_1, L_2 CFLs *implies* $L_1 \bullet L_2$ *is a* CFL.

Theorem 7.5.
CFLs *are closed under Kleene star.*
If L *is a* CFL $\implies$ L^* *is a* CFL.

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Closure Properties of CFLs

Union

$G_1 = (V_1, T, P_1, S_1)$ and $G_2 = (V_2, T, P_2, S_2)$
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Closure Properties of CFLs

Concatenation

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Closure Properties of CFLs

Stardom (i.e, Kleene star)

Theorem 7.8.

CFLs *are closed under Kleene star.*
If L *is a* CFL $\implies$ L^* *is a* CFL.

Exercise

- Prove that every regular language is context-free using previous closure properties.
- Prove the set of regular expressions over an alphabet $\mathbf{\Sigma}$ forms a non-regular language which is context-free.

Even more bad news: CFL not closed under complement

Theorem 7.9.

CFLs *are* *not* *closed under complement.*

Good news: Closure Properties of CFLs continued

Theorem 7.10.

If L_1 is a CFL and L_2 is regular then $L_1 \cap L_2$ is a CFL.

7.6.2
Parse trees and ambiguity

Parse Trees or Derivation Trees

A tree to represent the derivation $S \rightsquigarrow^* w$.

- Rooted tree with root labeled S
- Non-terminals at each internal node of tree
- Terminals at leaves
- Children of internal node indicate how non-terminal was expanded using a production rule

A picture is worth a thousand words

Parse Trees or Derivation Trees

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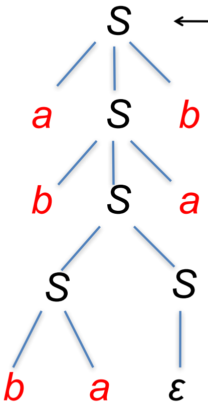
Example



← A derivation tree for *abbaab*

(also called “parse tree”)

$S \rightarrow aSb \mid bSa \mid SS \mid ab \mid ba \mid \varepsilon$

A corresponding derivation of *abbaab*

↓

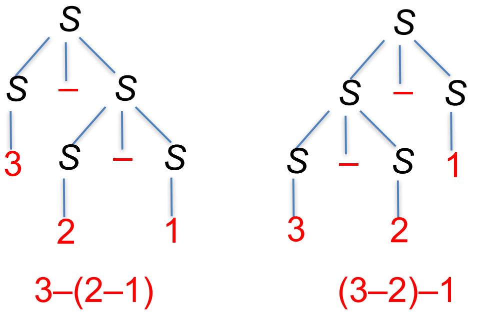
$S \Rightarrow aSb \Rightarrow abSab \Rightarrow abSSab \Rightarrow abbaSab \Rightarrow abbaab$

Ambiguity in CFLs

Definition 7.11.

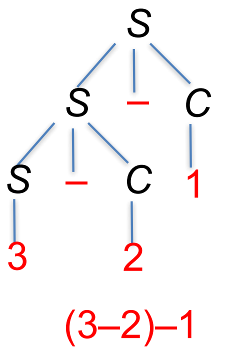
A CFG G is ambiguous if there is a string $w \in L(G)$ with two different parse trees. If there is no such string then G is unambiguous.

Example: $S \to S - S \mid 1 \mid 2 \mid 3$



Ambiguity in CFLs

- Original grammar: $S \to S - S \mid 1 \mid 2 \mid 3$
- Unambiguous grammar:
  $S \to S - C \mid 1 \mid 2 \mid 3$
  $C \to 1 \mid 2 \mid 3$



The grammar forces a parse corresponding to left-to-right evaluation.

Inherently ambiguous languages

Definition 7.12.

A CFL L is inherently ambiguous if there is no unambiguous CFG G such that $L = L(G)$.

- There exist inherently ambiguous CFLs.
  Example: $L = \{a^n b^m c^k \mid n = m \text{ or } m = k\}$
- Given a grammar G it is undecidable to check whether $L(G)$ is inherently ambiguous. No algorithm!

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7.7

CFGs; Proving a grammar generate a specific language

Inductive proofs for CFGs

Question: How do we formally prove that a CFG $L(G) = L$?

Example: $S \to \epsilon \mid a \mid b \mid aSa \mid bSb$

Theorem 7.1.

$L(G) = \{palindromes\} = \{w \mid w = w^R\}$

Two directions:

- $L(G) \subseteq L$, that is, $S \rightsquigarrow^* w$ then $w = w^R$
- $L \subseteq L(G)$, that is, $w = w^R$ then $S \rightsquigarrow^* w$

Inductive proofs for CFGs

Question: How do we formally prove that a CFG $L(G) = L$?

Example: $S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb$

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Two directions:

- $L(G) \subseteq L$, that is, $S \rightsquigarrow^* w$ then $w = w^R$
- $L \subseteq L(G)$, that is, $w = w^R$ then $S \rightsquigarrow^* w$

$L(G) \subseteq L$

Show that if $S \rightsquigarrow^* w$ then $w = w^R$

By induction on length of derivation, meaning
For all $k \geq 1$, $S \rightsquigarrow^{*k} w$ implies $w = w^R$.

- If $S \rightsquigarrow^1 w$ then $w = \epsilon$ or $w = a$ or $w = b$. Each case $w = w^R$.
- Assume that for all $k < n$, that if $S \rightarrow^k w$ then $w = w^R$
- Let $S \rightsquigarrow^n w$ (with $n > 1$). Wlog w begin with a.
  - Then $S \rightarrow aSa \rightsquigarrow^{k-1} aua$ where $w = aua$.
  - And $S \rightsquigarrow^{n-1} u$ and hence IH, $u = u^R$.
  - Therefore $w^r = (aua)^R = (ua)^R a = au^R a = aua = w$.

$L(G) \subseteq L$

Show that if $S \rightsquigarrow^* w$ then $w = w^R$

By induction on length of derivation, meaning
For all $k \geq 1$, $S \rightsquigarrow^{*k} w$ implies $w = w^R$.

- If $S \rightsquigarrow^1 w$ then $w = \epsilon$ or $w = a$ or $w = b$. Each case $w = w^R$.
- Assume that for all $k < n$, that if $S \rightarrow^k w$ then $w = w^R$
- Let $S \rightsquigarrow^n w$ (with $n > 1$). Wlog w begin with a.
  - Then $S \rightarrow aSa \rightsquigarrow^{k-1} aua$ where $w = aua$.
  - And $S \rightsquigarrow^{n-1} u$ and hence IH, $u = u^R$.
  - Therefore $w^r = (aua)^R = (ua)^R a = au^R a = aua = w$.

$L \subseteq L(G)$

Show that if $w = w^R$ then $S \rightsquigarrow^* w$.

By induction on $|w|$
That is, for all $k \geq 0$, $|w| = k$ and $w = w^R$ implies $S \rightsquigarrow^* w$.

Exercise: Fill in proof.

Mutual Induction

Situation is more complicated with grammars that have multiple non-terminals.

See Section 5.3.2 of the notes for an example proof.

7.8
CFGs normal form

Normal Forms

Normal forms are a way to restrict form of production rules

Advantage: Simpler/more convenient algorithms and proofs

Two standard normal forms for CFGs

- Chomsky normal form
- Greibach normal form

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Chomsky Normal Form:

- Productions are all of the form $A \rightarrow BC$ or $A \rightarrow a$.
  If $\epsilon \in L$ then $S \rightarrow \epsilon$ is also allowed.
- Every CFG G can be converted into CNF form via an efficient algorithm
- Advantage: parse tree of constant degree.

Greibach Normal Form:

- Only productions of the form $A \rightarrow a\beta$ are allowed.
- All CFLs without ϵ have a grammar in GNF. Efficient algorithm.
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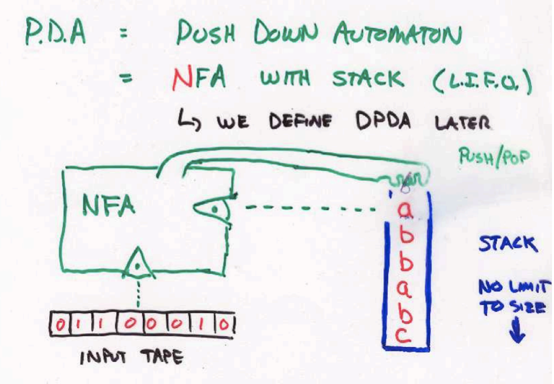
7.9
Pushdown automatas

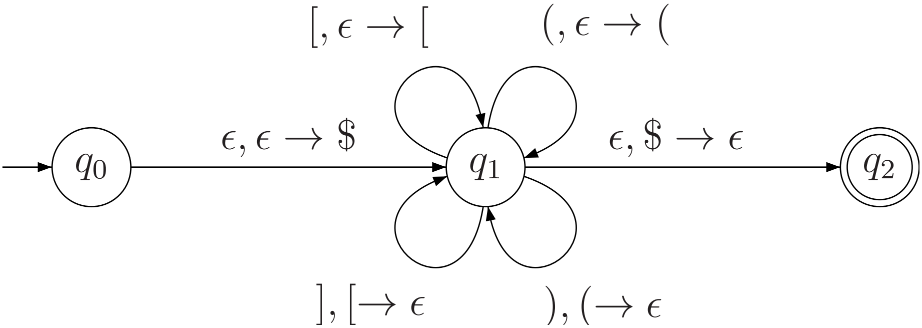
Things to know: Pushdown Automata

PDA: a NFA coupled with a stack



PDAs and CFGs are equivalent: both generate exactly CFLs.
PDA is a machine-centric view of CFLs.

Pushdown automata by example



7.10
Supplemental: Why $a^n b^n c^n$ is not CFL

You are bound to repeat yourself...

$L = \{a^n b^n c^n \mid n \geq 0\}$.

1. For the sake of contradiction assume that there exists a grammar: G a CFG for L.
2. T_i: **minimal** parse tree in G for $a^i b^i c^i$.
3. $h_i = \text{height}(T_i)$: Length of longest path from root to leaf in T_i.
4. For any integer t, there must exist an index $j(t)$, such that $h_{j(t)} > t$.
5. There an index j, such that $h_j > \left(2 * \#\text{ variables in } G\right)$.

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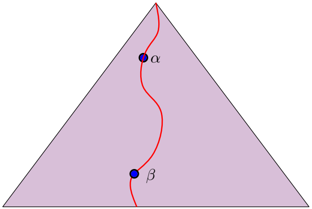
Repetition in the parse tree...



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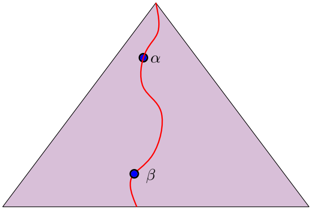

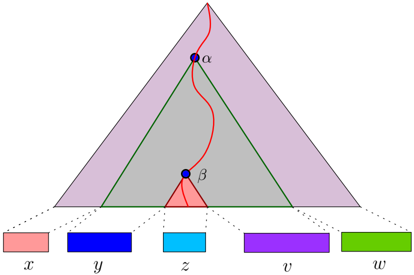

$xyzvw = a^j b^j c^j$

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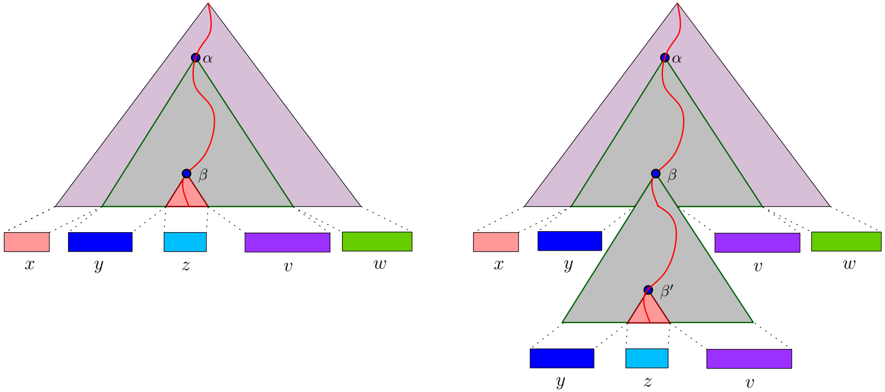

$$xyzvw = a^j b^j c^j \implies xy^2zv^2w \in L$$

Now for some case analysis...

- We know:
  $xyzvw = a^j b^j c^j$
  $|y| + |v| > 0$.
- We proved that $\tau = xy^2zv^2w \in L$.
- If y contains both a and b, then, $\tau = ...a...b...a...b...$.
  Impossible, since $\tau \in L = \{a^n b^n c^n \mid n \geq 0\}$.
- Similarly, not possible that y contains both b and c.
- Similarly, not possible that v contains both a and b.
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- If y contains only as, and v contains only bs, then... $\#_a(\tau) \neq \#_c(\tau)$.
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We conclude...

Lemma 7.1.

The language $L = \{a^n b^n c^n \mid n \geq 0\}$ *is not* CFL *(i.e., there is no* CFG *for it).*