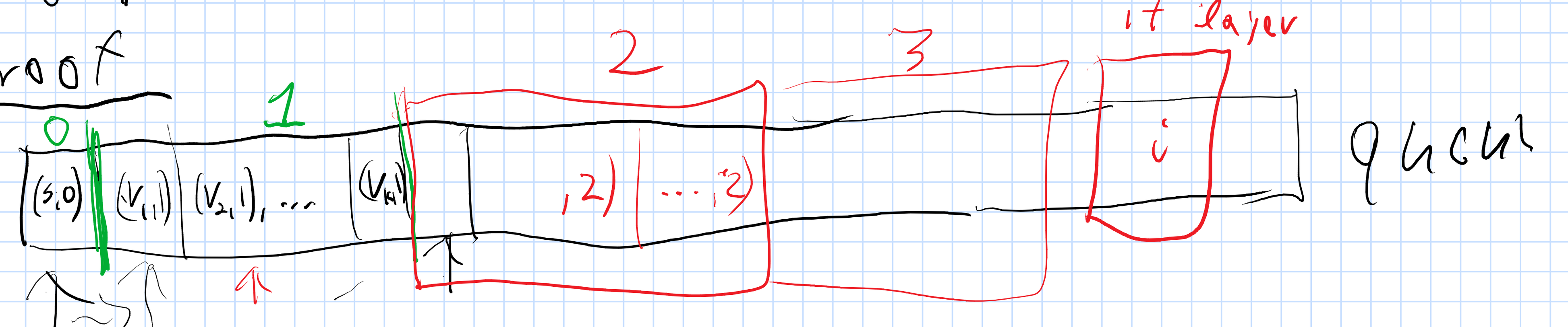
L18: Shortest Paths I (BFS and Dijkstra) 10/29/24

Lemma

For BFS the values of dist(v) are filled in increasing order.

proof



$s \to v_1, \dots, s \to v_k$ all outgoing edges from s. ∎

Lemma

For all $v \in V(G)$ dist(v) is indeed the length of the shortest path from s to v.

Proof

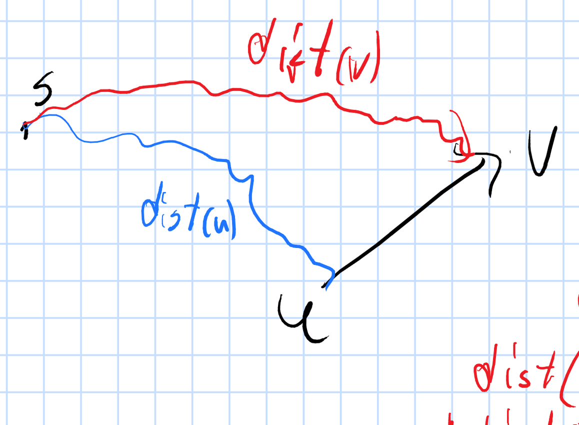
dist(s)=0; Base of induction.

I.H.: $\forall v \in V$ s.t. $\underbrace{d}_{\text{real distance}}(s,v) \le k$, dist(v) is computed correctly.

Inductive step: Let $v \in V$ s.t. $d(s,v) = k+1$.
Let π be the shortest path from s to v,

π: $s \to v_1 \to v_2 \cdots \to u \to v$

u is the predecessor of v, and as such
$d(s,u) = k \Rightarrow \text{dist}(u) = k$.
When u is handled by the algorithm, then it would schedule v, with $\text{dist}(v) \leftarrow k+1$.
If some other neighbor of v, say x is the first to be dequeued, then it must be $\text{dist}(x) = k$ and the above argument applies. ∎



triangle inequality

$\text{dist}(v) \le \text{dist}(u) + 1$

$\text{dist}(v) - \text{dist}(u) \le 1$

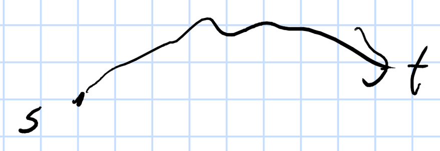
$|\text{dist}(v) - \text{dist}(u)| \le 1$ (undirected case)

O(n+m)

$G = (V, E)$

$\forall e \in E \quad w(e) > 0$

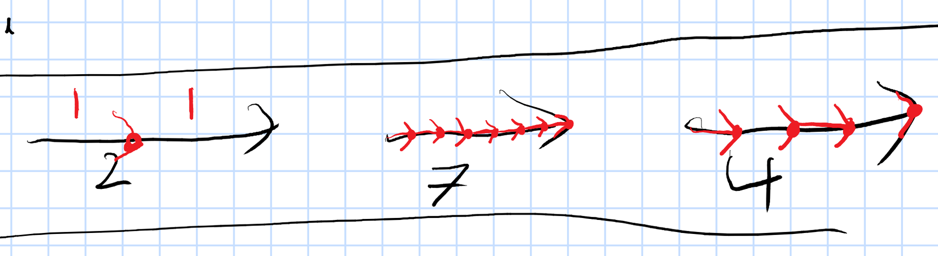
s: source

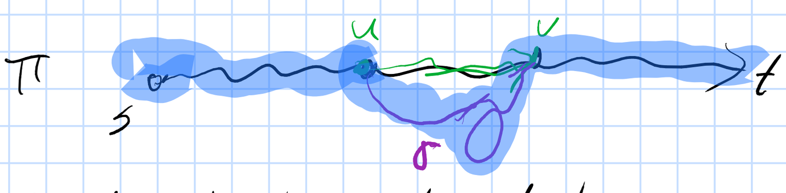


shortest path

$\Pi_{s,t}$: all paths from s to t

$\underset{\text{shortest path distance}}{d(s,t)} = \min_{\pi \in \Pi_{s,t}} w(\pi)$

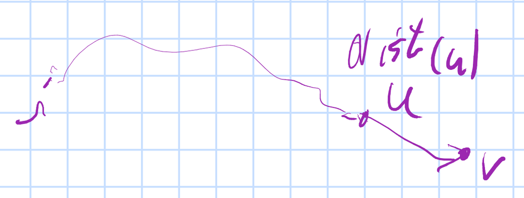




π: shortest path between s and t

u, v two internal vertices on π

$\Rightarrow$ shortest path between u and v is the subpath $\pi[u,v]$



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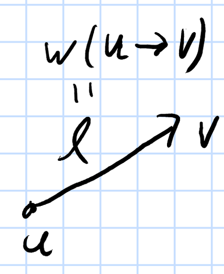
shortest_path_silly(G, s)
  dist(s) ← 0
  ∀v ∈ V − s dist(v) ← +∞
  while ∃ a tense edge e in G
    relax(e)
  return computed distances



(u,v) u→v, (u→v)

dist(u) current length of shortest path discovered from s to u

u→v is <u>tense</u> if

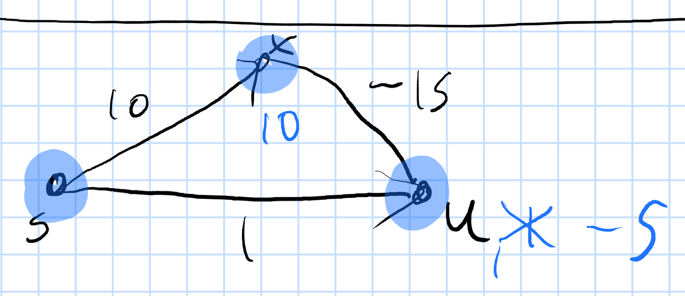
$\text{dist}(u) + \ell(u \to v) < \text{dist}(v)$

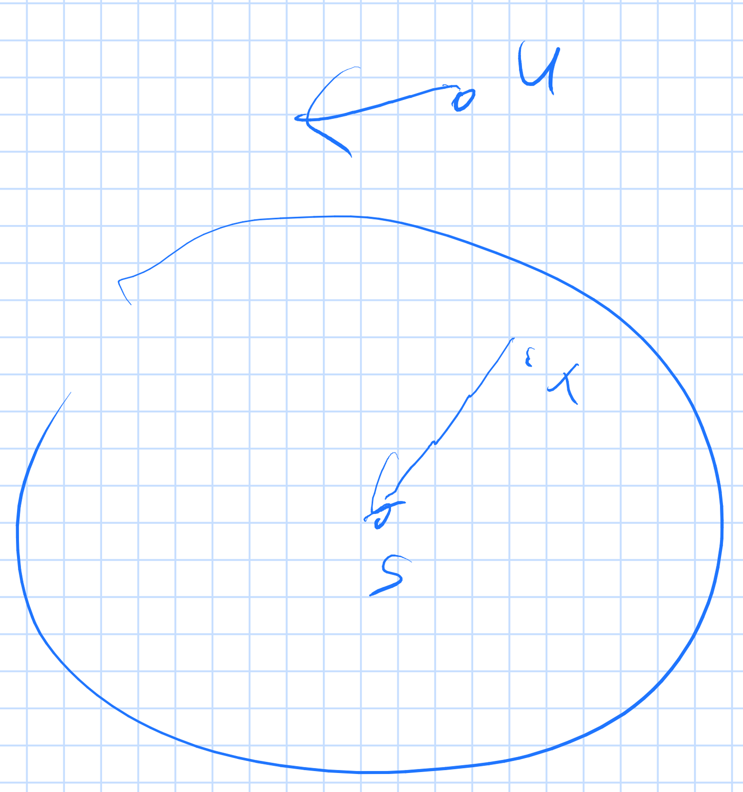
$\Rightarrow \text{dist}(v) \leftarrow \text{dist}(u) + \ell(u \to v)$ <u>relax</u>

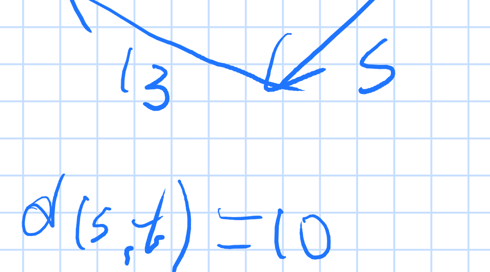
Fibonacci heaps

Priority queue:
- Insert $O(\log n)$
- extractMin $O(\log n)$ time
- decreaseKey $O(1)$ time (amortized)

$O(n \log n + m)$ Running time of Dijkstra







$d(s,t) = 10$

$d(t,s) = 18$

For undirected graph, with positive weights on the edges, the shortest path distance is a <u>metric</u>:

- $\forall v \in V \quad d(v,v) = 0$
- $\forall s,t \in V \quad d(s,t) = 0 \Rightarrow s = t$
- $\forall x,y,z \in V \quad d(x,y) + d(y,z) \ge d(x,z)$ [Triangle inequality]