Algorithms & Models of Computation

CS/ECE 374, Fall 2020

NP and NP Completeness

Lecture 23

Tuesday, December 1, 2020

LaTeXed: October 27, 2020 13:58

23.1
NP-Completeness: Cook-Levin Theorem

23.1.1
Completeness

NP: Non-deterministic polynomial

Definition 23.1.

A decision problem is in **NP**, *if it has a polynomial time certifier, for all the all the YES instances.*

Definition 23.2.

A decision problem is in **co-NP**, *if it has a polynomial time certifier, for all the all the NO instances.*

Example 23.3.

1. **3SAT** is in **NP**.
2. But **Not3SAT** is in **co-NP**.

In the beginning...

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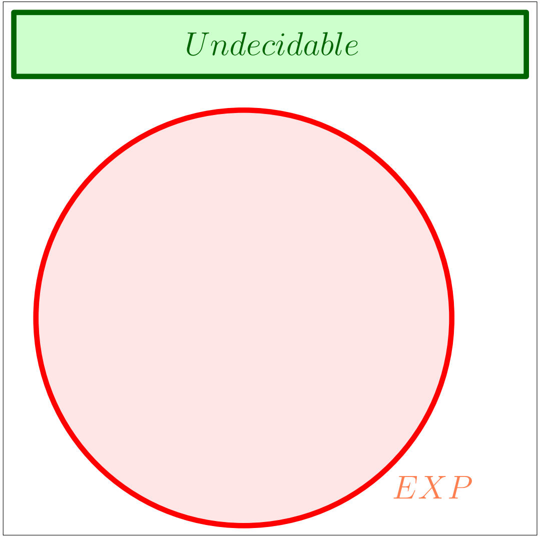

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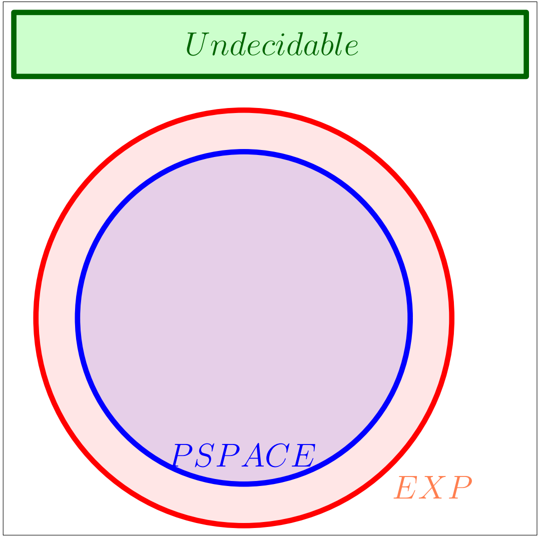

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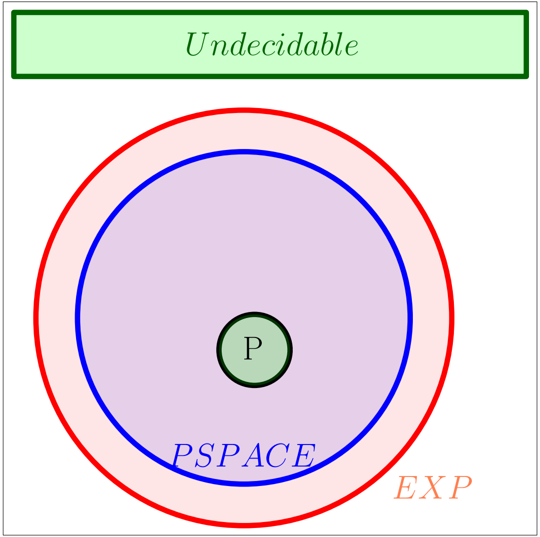

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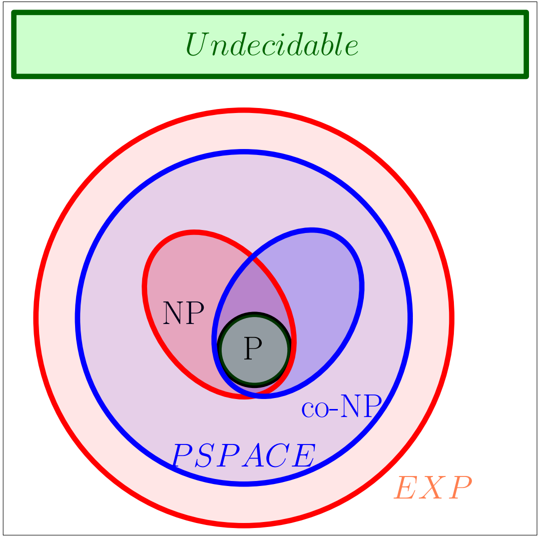

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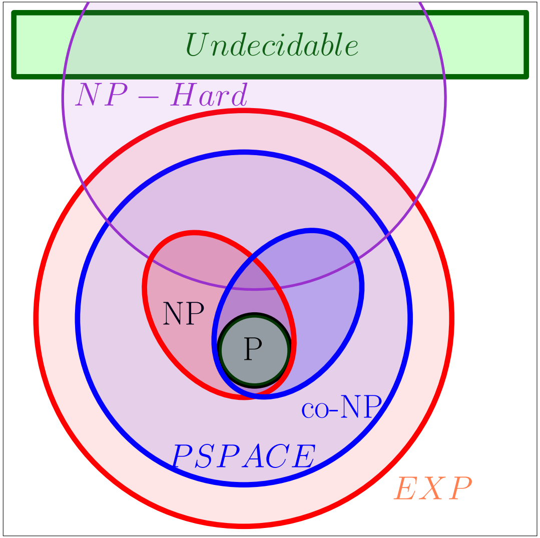

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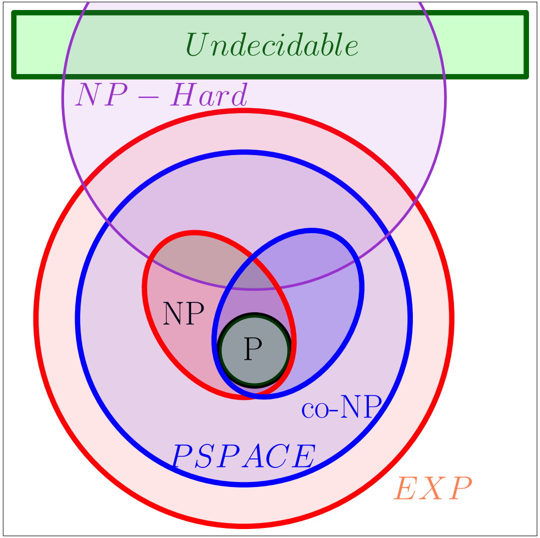

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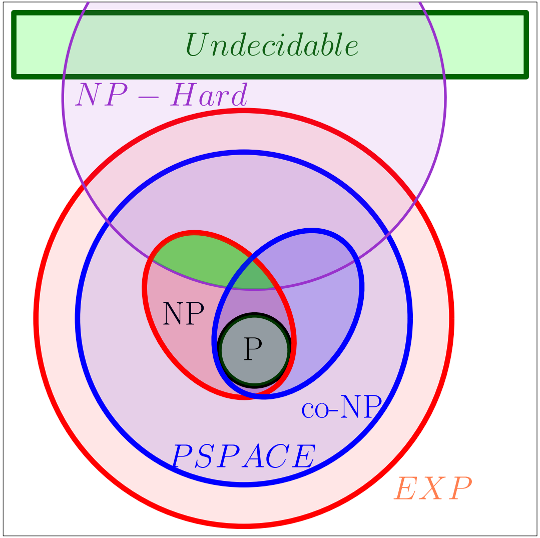

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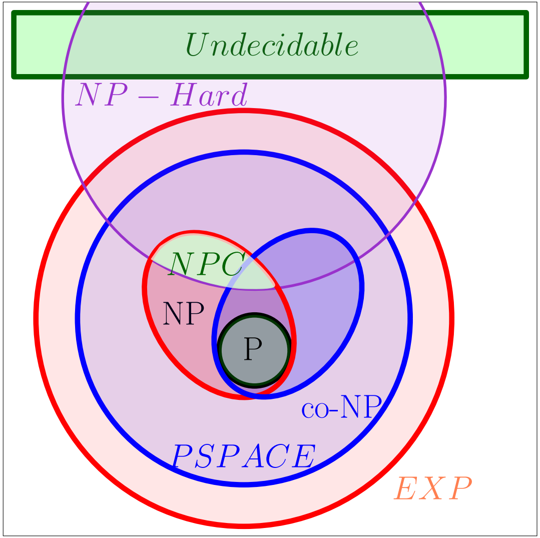

“Hardest” Problems

Question

What is the hardest problem in **NP**? How do we define it?

Towards a definition

1. Hardest problem must be in **NP**.
2. Hardest problem must be at least as “difficult” as every other problem in **NP**.

NP-Complete Problems

Definition 23.4.

A problem X is said to be **NP-Complete** if

1. $X \in \mathbf{NP}$, and
2. (Hardness) For any $Y \in \mathbf{NP}$, $\mathbf{Y} \leq_P \mathbf{X}$.

Solving **NP-Complete** Problems

Proposition 23.5.

Suppose $\boldsymbol{X}$ is **NP-Complete**. *Then $\boldsymbol{X}$ can be solved in polynomial time* $\iff$ **P** $=$ **NP**.

Proof.

$\Rightarrow$ Suppose $\boldsymbol{X}$ can be solved in polynomial time

0.1 Let $\boldsymbol{Y} \in$ **NP**. We know $\mathbf{Y} \leq_P \mathbf{X}$.

0.2 We showed that if $\mathbf{Y} \leq_P \mathbf{X}$ and $\boldsymbol{X}$ can be solved in polynomial time, then $\boldsymbol{Y}$ can be solved in polynomial time.

0.3 Thus, every problem $\boldsymbol{Y} \in$ **NP** is such that $\boldsymbol{Y} \in \boldsymbol{P}$.

0.4 $\implies \boldsymbol{NP} \subseteq \boldsymbol{P}$.

0.5 Since **P** $\subseteq$ **NP**, we have **P** $=$ **NP**.

$\Leftarrow$ Since **P** $=$ **NP**, and $\boldsymbol{X} \in$ **NP**, we have a polynomial time algorithm for $\boldsymbol{X}$. □

NP-Hard Problems

Definition 23.6.

A problem X is said to be **NP-Hard** if

1. (Hardness) For any $Y \in$ **NP**, we have that $\mathbf{Y} \leq_P \mathbf{X}$.

An **NP-Hard** problem need not be in **NP**!

Example: Halting problem is **NP-Hard** (why?) but not **NP-Complete**.

Consequences of proving **NP-Completeness**

If X is **NP-Complete**

1. Since we believe $\mathbf{P} \neq \mathbf{NP}$,
2. and solving X implies $\mathbf{P} = \mathbf{NP}$.

X is unlikely to be efficiently solvable.

At the very least, many smart people before you have failed to find an efficient algorithm for X.
(This is proof by mob opinion — take with a grain of salt.)

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THE END

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(for now)