20.4
Correctness of the MST algorithms

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Safe edges must be in the MST

Correctness of MST Algorithms

1. Many different MST algorithms
2. All of them rely on some basic properties of MSTs, in particular the **Cut Property** to be seen shortly.

Key Observation: Cut Property

Lemma 20.1.

If e is a safe edge then **every** *minimum spanning tree contains e.*

Proof.

1. Suppose (for contradiction) e is not in MST T.
2. Since e is safe there is an $S \subset V$ such that e is the unique min cost edge crossing S.
3. Since T is connected, there must be some edge f with one end in S and the other in $V \setminus S$
4. Since $c_f > c_e$, $T' = (T \setminus \{f\}) \cup \{e\}$ is a spanning tree of lower cost! Error: T' may not be a spanning tree!!

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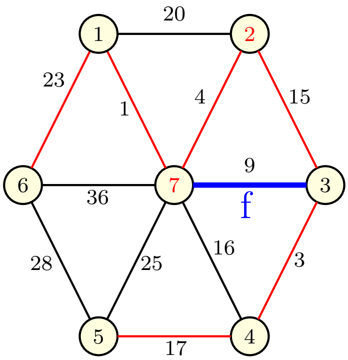
Error in Proof: Example

Problematic example. $S = \{1, 2, 7\}$, $e = (7, 3)$, $f = (1, 6)$. $T - f + e$ is not a spanning tree.



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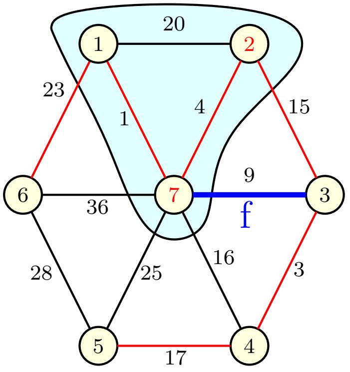

(B)

(A) Consider adding the edge ***f***.

(B) It is safe because it is the cheapest edge in the cut.

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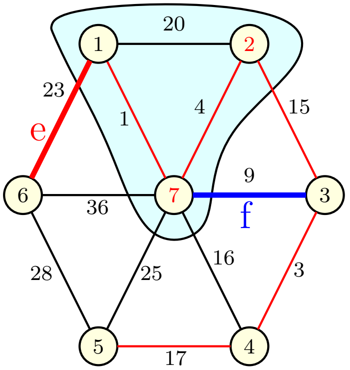

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(C) Lets throw out the edge ***e*** currently in the spanning tree which is more expensive than ***f*** and is in the same cut. Put it ***f*** instead...

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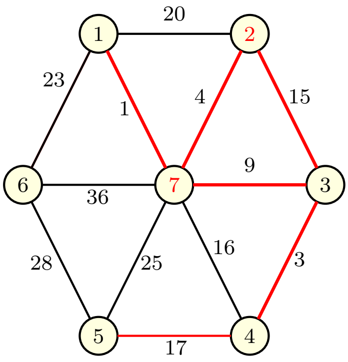

(D)

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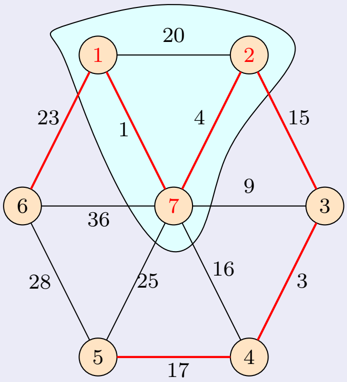
(B) It is safe because it is the cheapest edge in the cut.

(C) Lets throw out the edge e currently in the spanning tree which is more expensive than f and is in the same cut. Put it f instead...

(D) New graph of selected edges is not a tree anymore. BUG.

Proof of Cut Property

Proof.



1. Suppose $e = (v, w)$ is not in MST T and e is min weight edge in cut $(S, V \setminus S)$. Assume $v \in S$.
2. T is spanning tree: there is a unique path P from v to w in T
3. Let w' be the first vertex in P belonging to $V \setminus S$; let v' be the vertex just before it on P, and let $e' = (v', w')$
4. $T' = (T \setminus \{e'\}) \cup \{e\}$ is spanning tree of lower cost. (Why?) □

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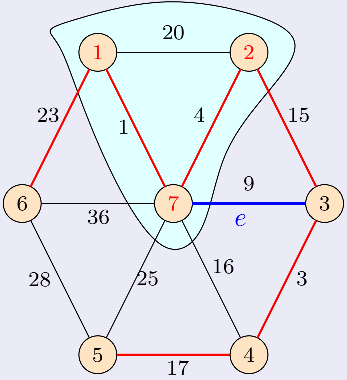

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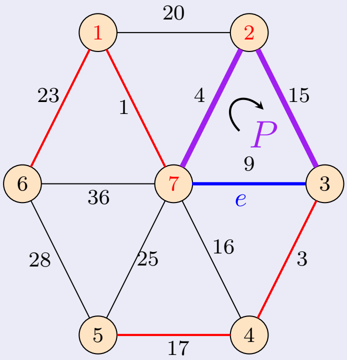

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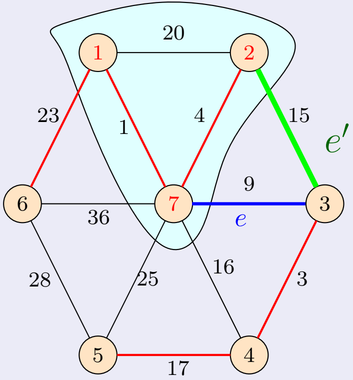

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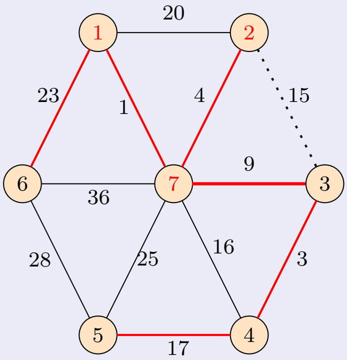

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Proof of Cut Property (contd)

Observation 20.2.

$T' = (T \setminus \{e'\}) \cup \{e\}$ *is a spanning tree.*

Proof.

T' is connected.

Removed $e' = (v', w')$ from T but v' and w' are connected by the path $P - f + e$ in T'. Hence T' is connected if T is.

T' is a tree

T' is connected and has $n - 1$ edges (since T had $n - 1$ edges) and hence T' is a tree

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THE END

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