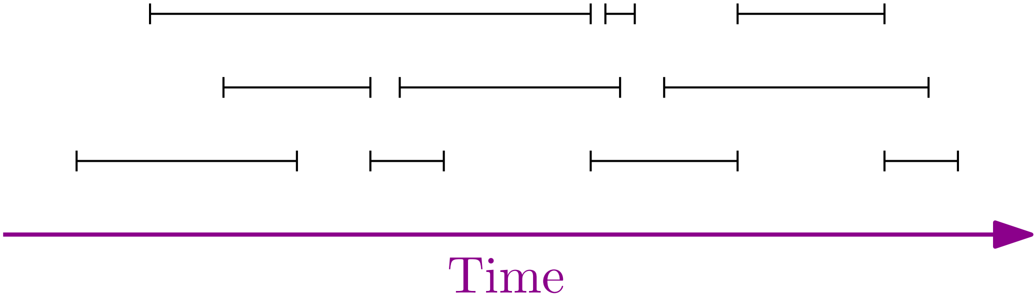
19.6.3
Proving optimality of earliest finish time

Earliest finish time: A quick recall



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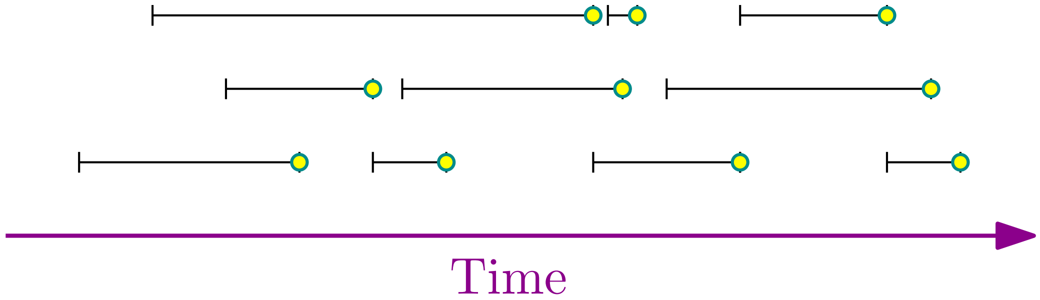

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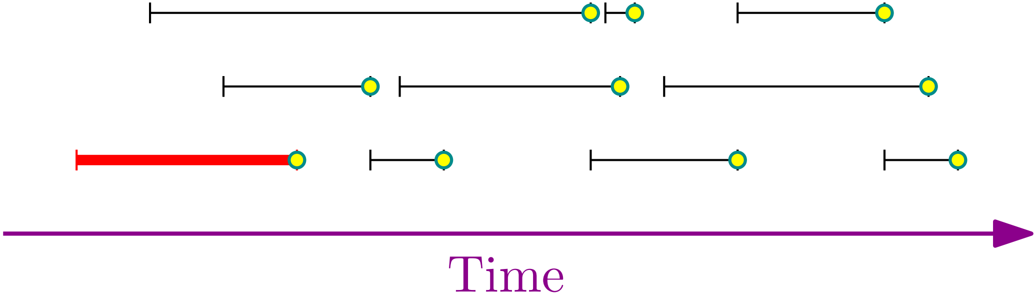

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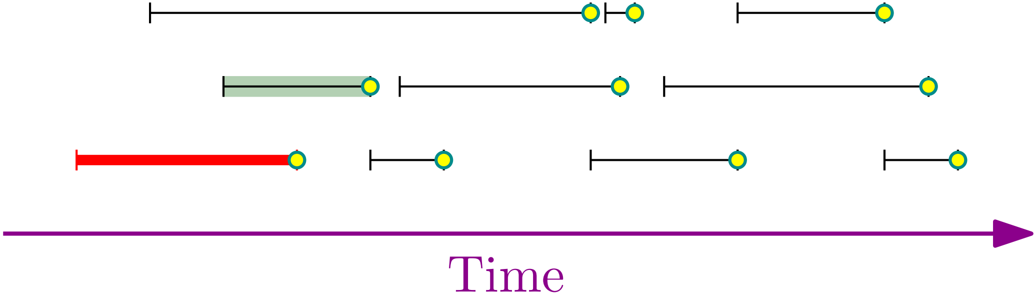

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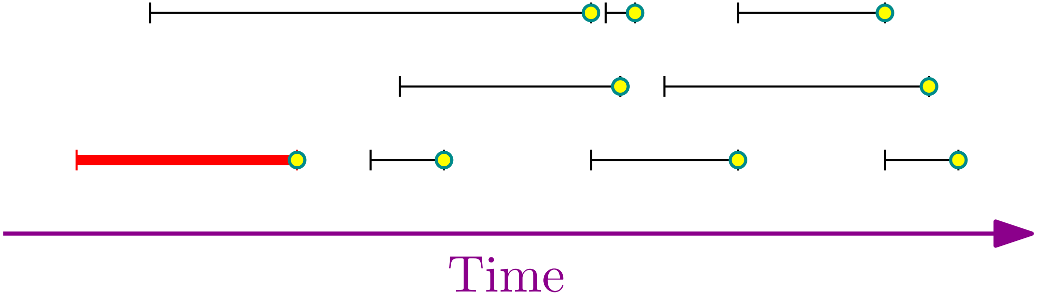

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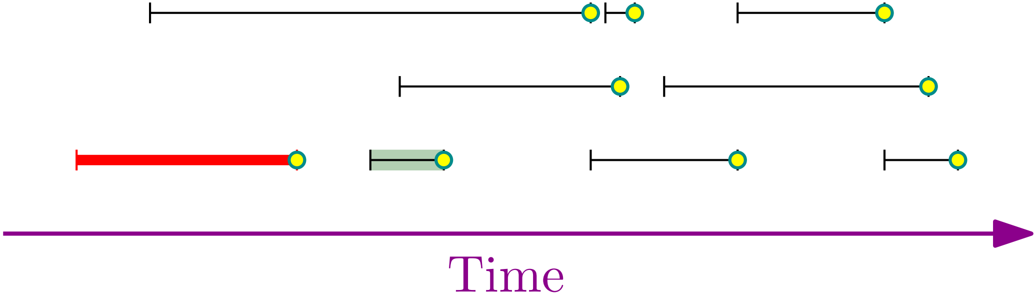

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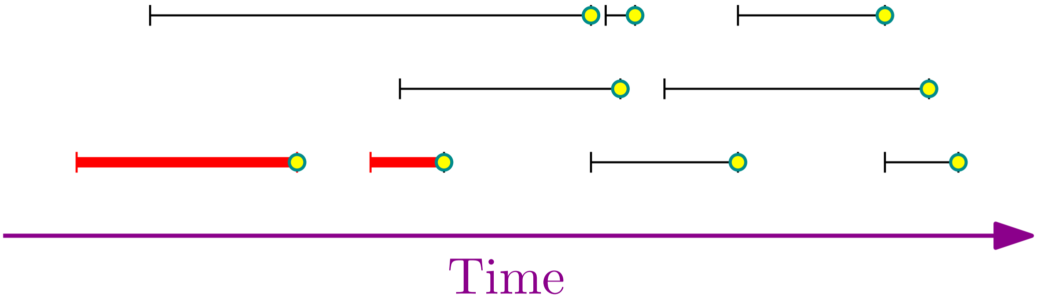

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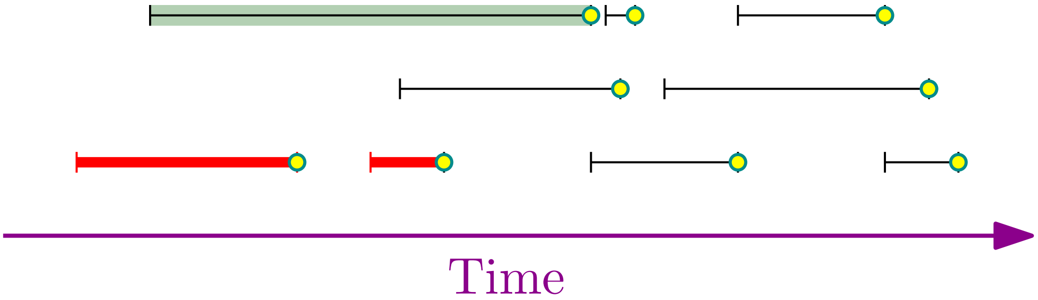

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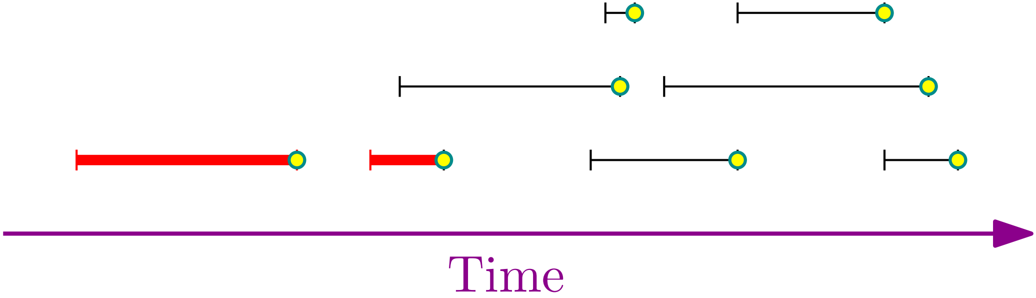

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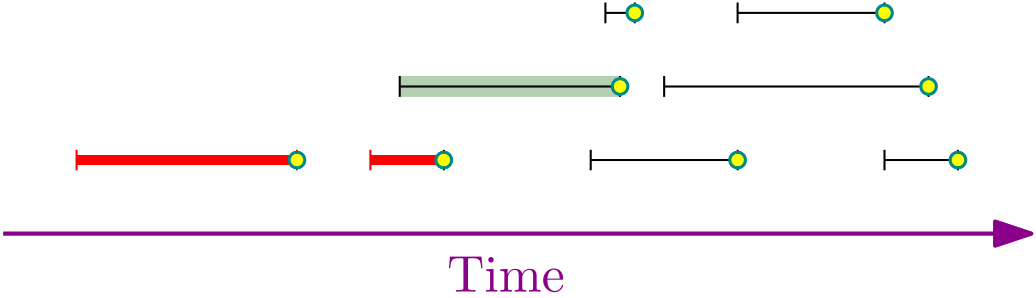

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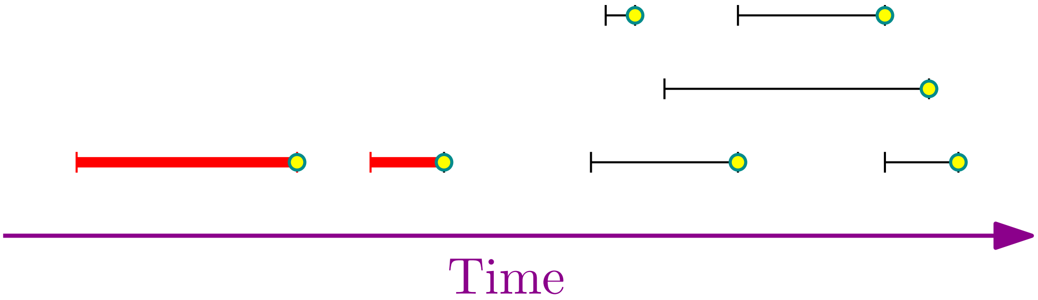

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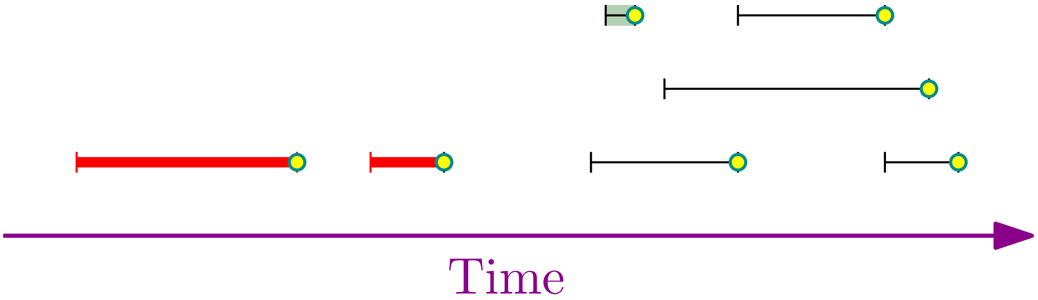

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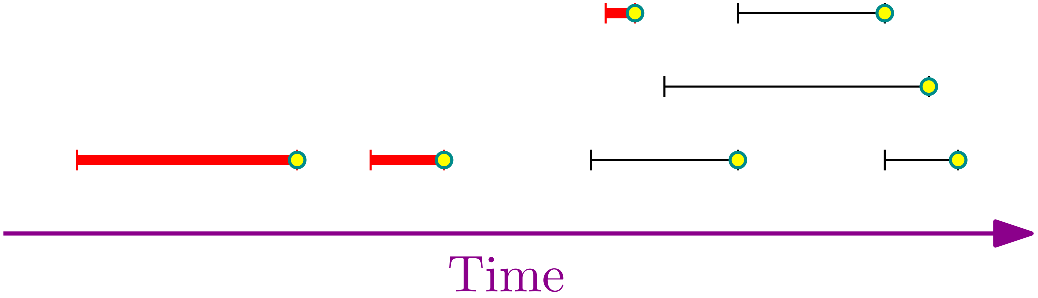

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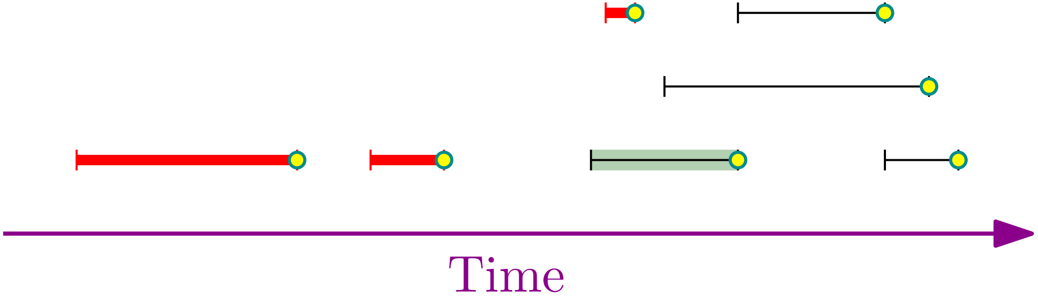

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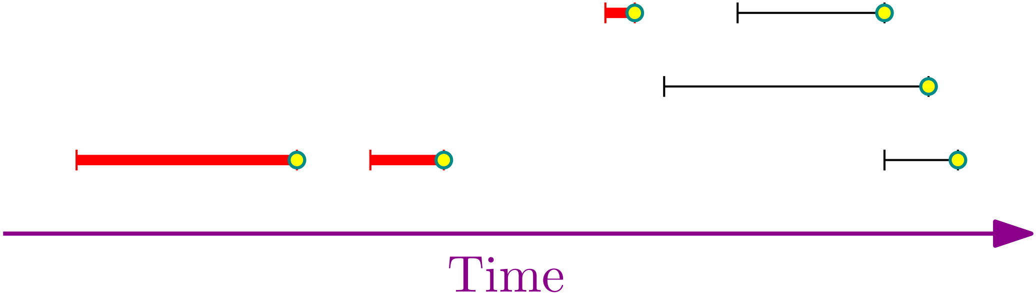

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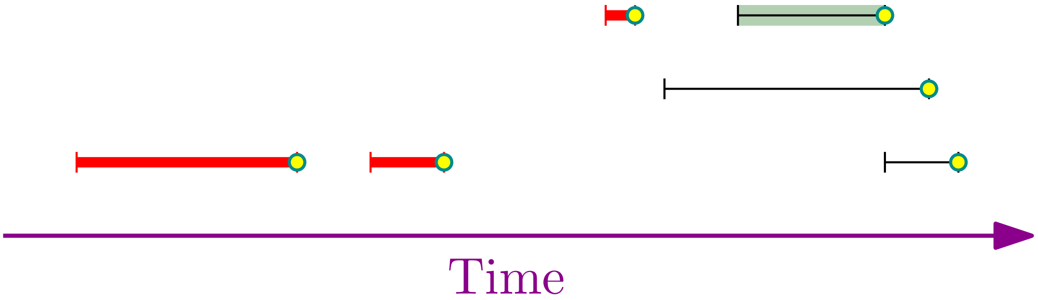

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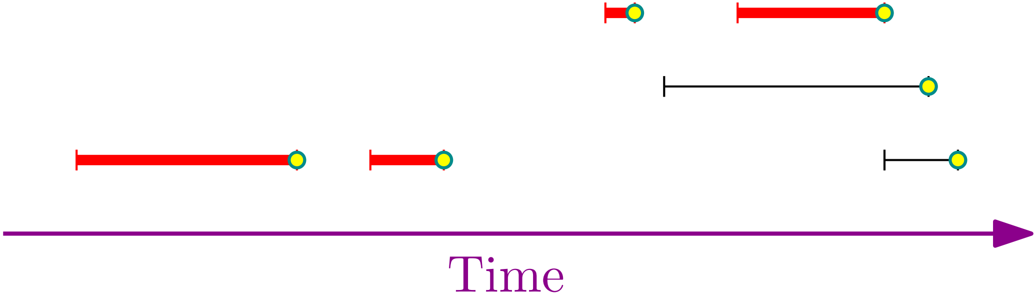

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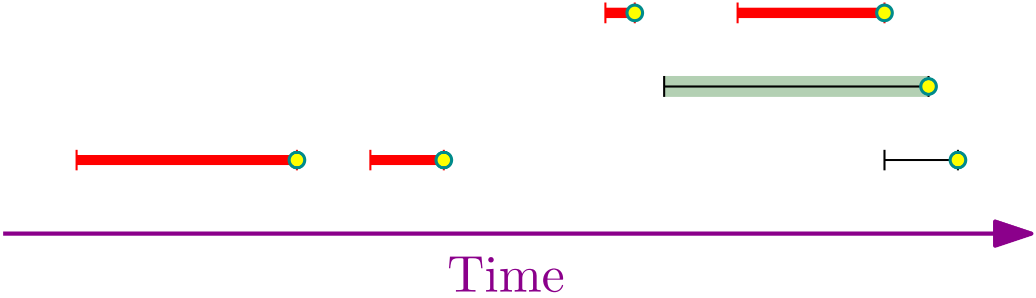

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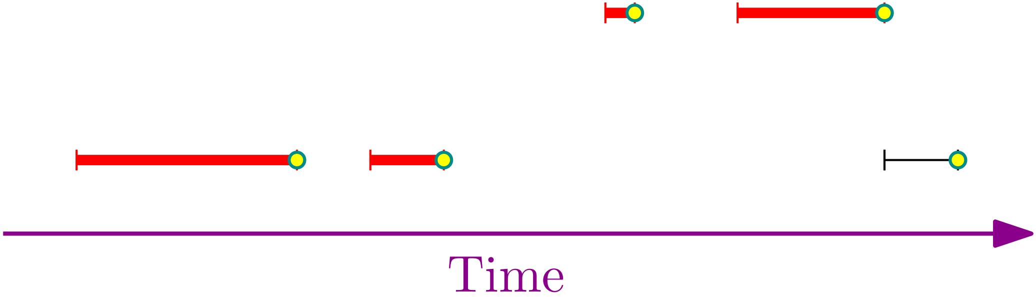

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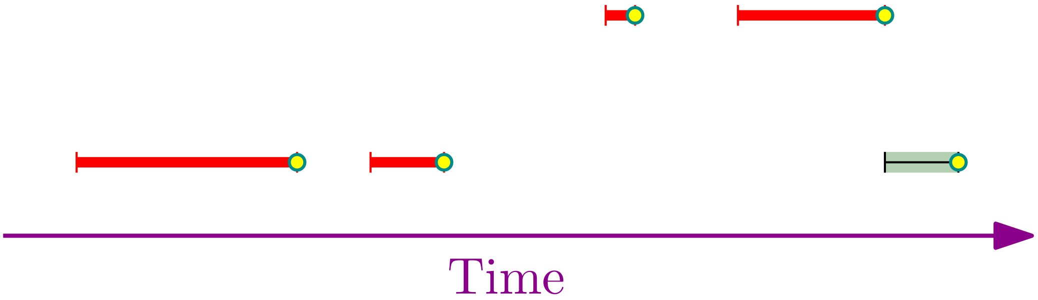

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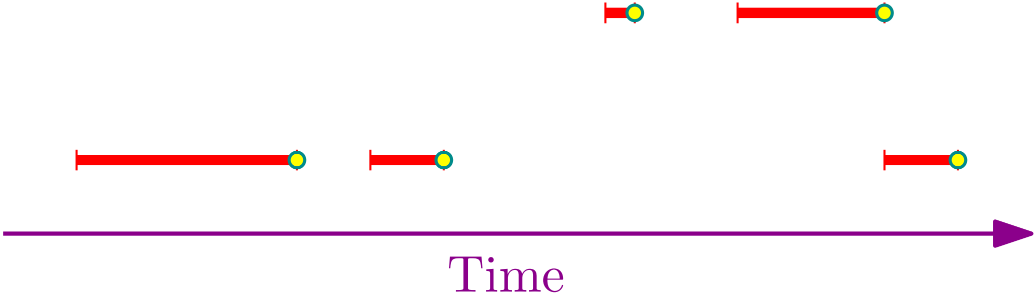

Proving Optimality

1. Correctness: Clearly the algorithm returns a set of jobs that does not have any conflicts
2. For a set of requests R, let O be an optimal set and let X be the set returned by the greedy algorithm. Then $O = X$? Not likely!

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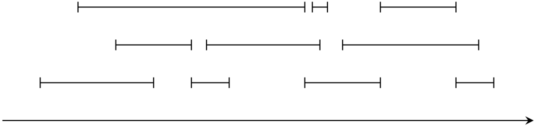

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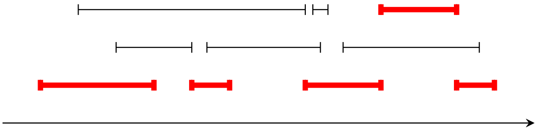

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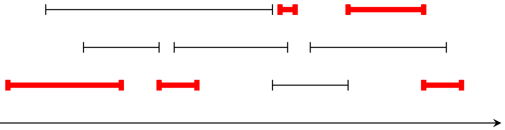

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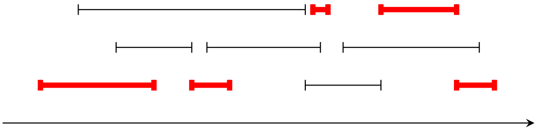

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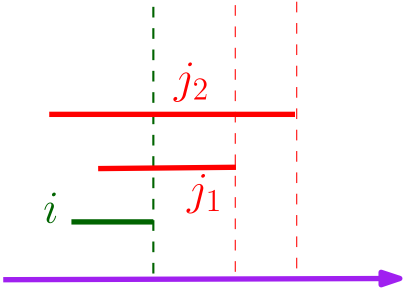
Helper Claim

Claim 19.3.

i be first interval picked by Greedy into solution.
O: Optimal solution.
If $i \notin O$, there is exactly one interval $j_1 \in O$ that conflicts with i.

Proof.

1. No $j \in O$ conflicts $i \implies O$ is not opt!
2. Suppose $j_1, j_2 \in O$ such that $j_1 \neq j_2$ and both j_1 and j_2 conflict with i.
3. Since i has earliest finish time, j_1 and i overlap at $f(i)$.
4. For same reason j_2 also overlaps with i at $f(i)$.
5. Implies that j_1, j_2 overlap at $f(i)$ but intervals in O cannot overlap. □



Proof of Optimality: Key Lemma

Lemma 19.4.

i_1 be first interval picked by Greedy. There exists an optimum solution that contains i_1.

Proof.

Let O be an arbitrary optimum solution. If $i_1 \in O$ we are done.

By **Claim 19.3** ...

1. Exists exactly one $j_1 \in O$ conflicting with i_1.
2. Form a new set O' by removing j_1 from O and adding i_1, that is $O' = (O - \{j_1\}) \cup \{i_1\}$.
3. From claim, O' is a feasible solution (no conflicts).
4. Since $|O'| = |O|$, O' is also an optimum solution and it contains i_1. □

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Proof of Optimality of Earliest Finish Time First

Proof by Induction on number of intervals.

Base Case: $n = 1$. Trivial since Greedy picks one interval.

Induction Step: Assume theorem holds for $i < n$.

Let K be an input (i.e., instance) with n intervals

$i_1 \Leftarrow$ First interval picked by greedy algorithm.

$K' \Leftarrow$ The result of removing i_1 and all conflicting intervals from K.

$|K'| = |K| - 1$.

$G(K), G(K')$: Solution produced by Greedy on K and K', respectively.

Lemma 19.4 $\implies$ optimum solution O to K with $i_1 \in O$.

Let $O' = O - \{i_1\}$. O' is a solution to K'.

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\begin{aligned}
|G(K)| &= 1 + |G(K')| && \text{from Greedy description} \\
&\geq 1 + |O'| && \text{By induction, } G(I') \text{ is optimum for } I') \\
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