18.1.4
Applications of shortest path for negative weights on edges

Why negative lengths?

Several Applications

1. Shortest path problems useful in modeling many situations — in some negative lengths are natural
2. Negative length cycle can be used to find arbitrage opportunities in currency trading
3. Important sub-routine in algorithms for more general problem: minimum-cost flow

Negative cycles

Application to Currency Trading

Currency Trading

Input: n currencies and for each ordered pair (a, b) the exchange rate for converting one unit of a into one unit of b.

Questions:

1. Is there an arbitrage opportunity?
2. Given currencies s, t what is the best way to convert s to t (perhaps via other intermediate currencies)?

Concrete example:

1. 1 Chinese Yuan $= 0.1116$ Euro
2. 1 Euro $= 1.3617$ US dollar
3. 1 US Dollar $= 7.1$ Chinese Yuan.

Thus, if exchanging 1 \$ $\rightarrow$ Yuan $\rightarrow$ Euro $\rightarrow$ \$, we get: $0.1116 * 1.3617 * 7.1 = 1.07896$\$.

Reducing Currency Trading to Shortest Paths

Observation: If we convert currency i to j via intermediate currencies $k_1, k_2, \ldots, k_h$ then one unit of i yields $exch(i, k_1) \times exch(k_1, k_2) \ldots \times exch(k_h, j)$ units of j.

Create currency trading directed graph $G = (V, E)$:

1. For each currency i there is a node $v_i \in V$
2. $E = V \times V$: an edge for each pair of currencies
3. edge length $\ell(v_i, v_j) = -\log(exch(i, j))$ can be negative

Exercise: Verify that

1. There is an arbitrage opportunity if and only if G has a negative length cycle.
2. The best way to convert currency i to currency j is via a shortest path in G from i to j. If d is the distance from i to j then one unit of i can be converted into 2^d units of j.

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Create currency trading directed graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$:

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Exercise: Verify that

1. There is an arbitrage opportunity if and only if $\boldsymbol{G}$ has a negative length cycle.
2. The best way to convert currency $\boldsymbol{i}$ to currency $\boldsymbol{j}$ is via a shortest path in $\boldsymbol{G}$ from $\boldsymbol{i}$ to $\boldsymbol{j}$. If $\boldsymbol{d}$ is the distance from $\boldsymbol{i}$ to $\boldsymbol{j}$ then one unit of $\boldsymbol{i}$ can be converted into $\boldsymbol{2^d}$ units of $\boldsymbol{j}$.

Reducing Currency Trading to Shortest Paths

Math recall - relevant information

1. $\log(\alpha_1 * \alpha_2 * \cdots * \alpha_k) = \log \alpha_1 + \log \alpha_2 + \cdots + \log \alpha_k$.
2. $\log x > 0$ if and only if $x > 1$.

THE END

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(for now)