Dynamic Programming: Shortest Paths and DFA to Reg Expressions

Lecture 18

Thursday, October 29, 2020

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18.1
Shortest Paths with Negative Length Edges

18.1.1
Why Dijkstra's algorithm fails with negative edges

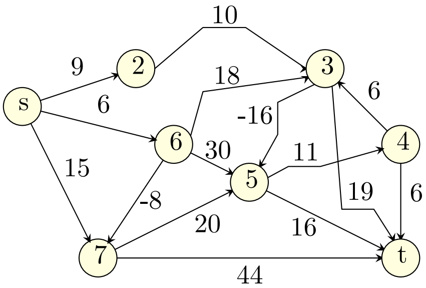
Single-Source Shortest Paths with Negative Edge Lengths

Problem statement

Single-Source Shortest Path Problems

Input: A directed graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes s, t find shortest path from s to t.
2. Given node s find shortest path from s to all other nodes.



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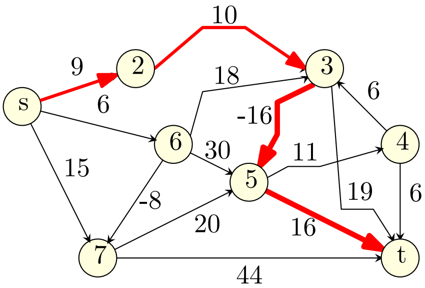
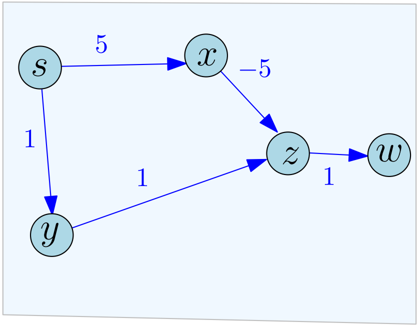


What are the distances computed by Dijkstra's algorithm?

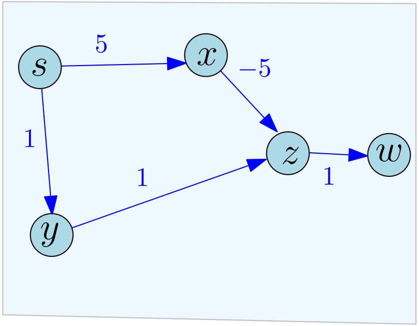


The distance as computed by Dijkstra algorithm starting from s:

- (A) $s = 0$, $x = 5$, $y = 1$, $z = 0$.
- (B) $s = 0$, $x = 1$, $y = 2$, $z = 5$.
- (C) $s = 0$, $x = 5$, $y = 1$, $z = 2$.
- (D) IDK.

Dijkstra's Algorithm and Negative Lengths

With negative length edges, Dijkstra's algorithm can fail



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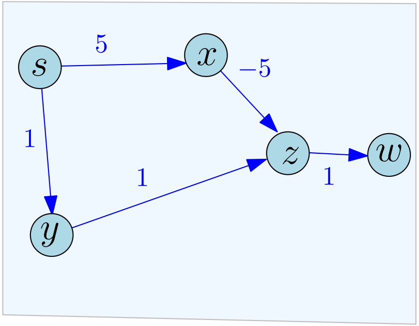

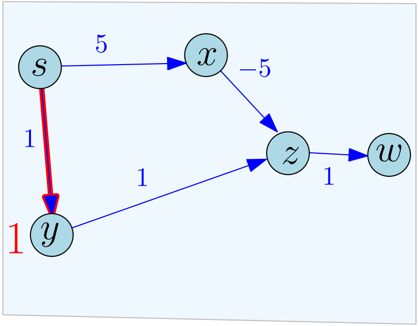

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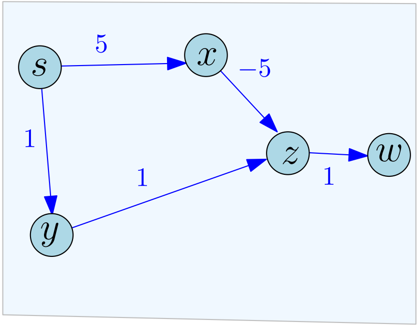

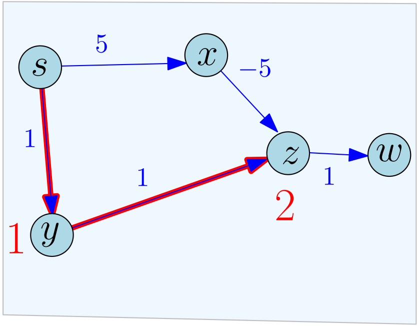

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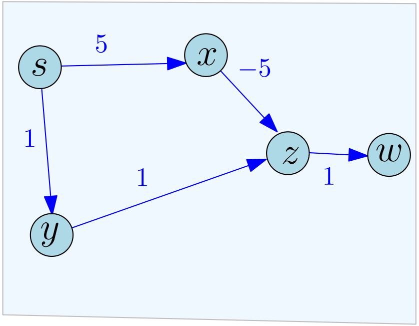

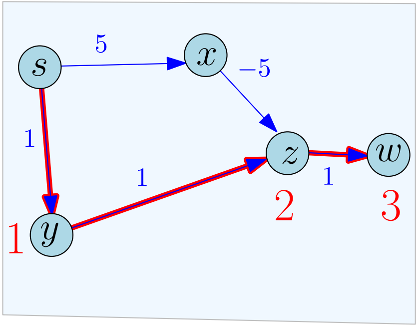

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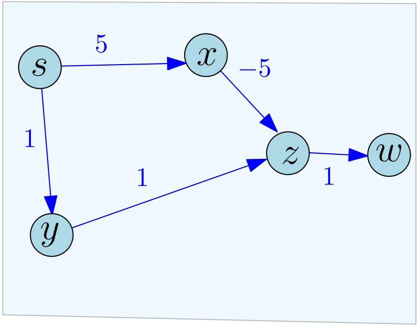

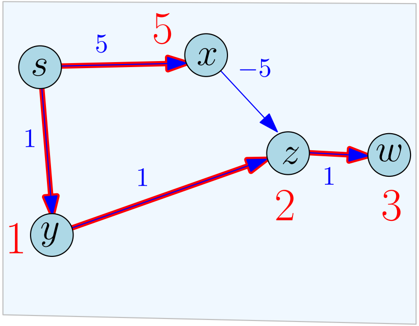

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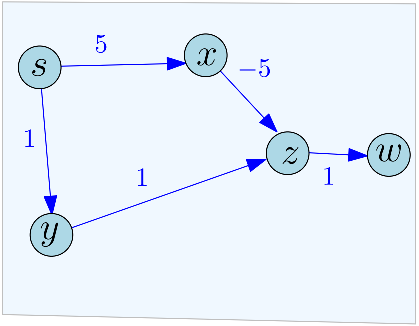

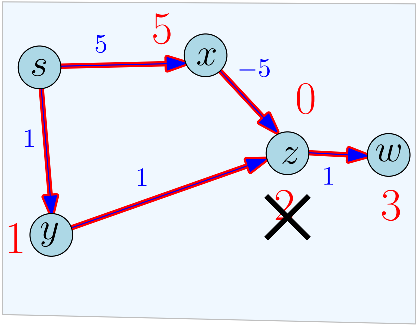

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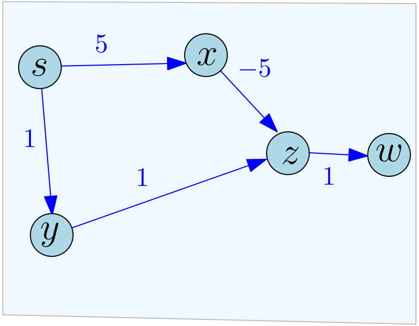

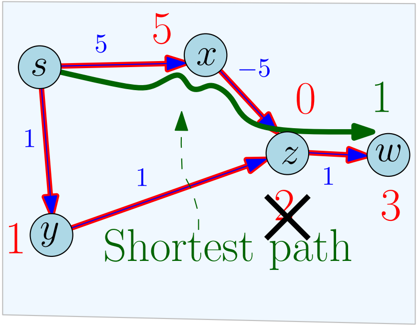

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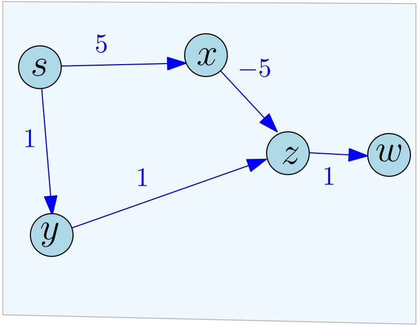

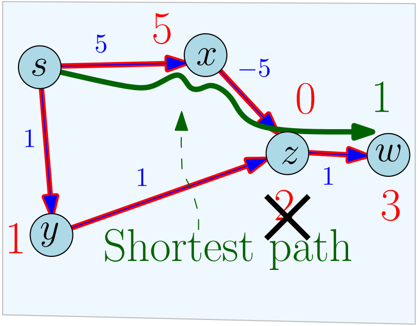

False assumption: Dijkstra's algorithm is based on the assumption that if $s = v_0 \to v_1 \to v_2 \ldots \to v_k$ is a shortest path from s to v_k then $dist(s, v_i) \leq dist(s, v_{i+1})$ for $0 \leq i < k$. Holds true only for non-negative edge lengths.

Shortest Paths with Negative Lengths

Lemma 18.1.

Let G be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

1. $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ *is a shortest path from* s *to* v_i
2. *False:* $\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$ *for* $1 \leq i < k$*. Holds true only for non-negative edge lengths.*

Cannot explore nodes in increasing order of distance! We need other strategies.

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THE END

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(for now)