13.7
Tangential: Fibonacci and his numbers

Fibonacci = Leonardo Bonacci

1. CE 1170–1250.
2. Italian. Spent time in Bugia, Algeria with his father (trader).
3. Traveled around the Mediterranean coast, learned the Hindu–Arabic numerals
4. Hindu–Arabic numerals:
    1. Developed before 400 CE by Hindu philosophers.
    2. Arrived to the Arab world sometime before 825CE.
    3. Muhammad ibn Musa al-Khwarizmi (Algorithm/Algebra) wrote a book in 825 CE explaining the new system. (Showed how to solved quadratic equations.)
5. 1202 CE: Fibonacci wrote a book "Liber Abaci" (book of calculations) that popularized the new system.
6. Brought and popularized the Hindu–Arabic system to Italy.

Fibonacci numbers

1. Fibonacci in Liber Abaci posed and solved a problem involving the growth of a population of rabbits based on idealized assumptions.
2. Describe growth processes.
   Every month a mature pair of rabbits give birth to one pair of young rabbits.

Month	grownup pairs	Young pairs
1	1	0
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2. Golden ratio: $\varphi = (\sqrt{5}+1)/2 \approx 1.618033$.
3. For $a > b > 0$, $\varphi = \frac{a+b}{a} = \frac{a}{b}$. $\implies$ $\frac{\varphi+1}{\varphi} = \varphi$. $\implies$ $0 = \varphi^2 - \varphi - 1$.
4. $\varphi = \frac{1\pm\sqrt{1+4}}{2}$ since φ is not negative, so...
5. $F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$
6. Golden ratio goes back to Euclid
7. Many applications of GR and Fibonacci numbers in nature, models (stock market), art, etc...

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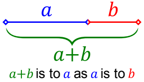

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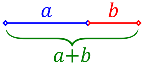

$a+b$ is to a as a is to b

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Fibonacci numbers: Binet's formula

1. $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi$ are solution to the equation: $x^2 = x + 1$.
2. As such, φ and ψ a solution to the equation: $x^n = x^{n-1} + x^{n-2}$.
3. Consider the sequence $U_n = U_{n-1} + U_{n-2}$.
   For any $\alpha, \beta \in \mathbb{R}$, consider $U_n = \alpha\varphi^n + \beta\psi^n$. A valid solution to the sequence.

$$U_n = U_{n-1} + U_{n-2} = \alpha\varphi^{n-1} + \beta\psi^{n-1} + \alpha\varphi^{n-2} + \beta\psi^{n-2}$$

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   $\implies \alpha = 1/\sqrt{5} \implies F_n = U_n = \alpha\varphi^n + \beta\psi^n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$

Fibonacci numbers really fast

$$\begin{pmatrix} y \\ x+y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

As such,

$$\begin{aligned} \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n-2} \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} F_{n-3} \\ F_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n-3} \begin{pmatrix} F_2 \\ F_1 \end{pmatrix}. \end{aligned}$$

More on fast Fibonacci numbers

Continued

Thus, computing the nth Fibonacci number can be done by computing $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n-3}$.

Which can be done in $O(\log n)$ time (how?). What is wrong?