9.4
Unrecognizable

TM recognizable

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Language L *is* TM ***recognizable*** *if there exists* M *that stops on some inputs, such that* $L(M) = L$.

Theorem (Halting)

$A_{\mathrm{TM}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a TM and } M \text{ accepts } w \right\}$. *is* TM *recognizable, but not decidable.*

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If L and $\overline{L} = \Sigma^* \setminus L$ are both TM recognizable, then L and $\overline{L}$ are decidable.

Proof.

M: TM recognizing L.
M_c: TM recognizing $\overline{L}$.
Given input x, using UTM simulating running M and M_c on x in parallel. One of them must stop and accept. Return result.
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Complement language for $\mathbf{A}_{\mathrm{TM}}$

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But don't really care about invalid inputs. So, really:

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THE END

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(for now)