1 You are given a set of T of m toys, and there are n children. The ith child has a set of toys $T_i \subseteq T$ that they are willing to play with. Decide if there is a way of giving a toy to each child, so that they are all happy (i.e., playing with a toy they like). Assuming the input size is $O(nm)$ (why?), what is the running time of your algorithm solving this problem?

Solution:

Build the natural bipartite graph, and compute the maximum matching. The natural graph has nm edges, and the maximum bipartite matching algorithm seen in class requires $O(n \cdot nm) = O(n^4)$ time in this case.

2 Prove Hall's theorem:

Theorem 0.1 (Hall's theorem). *For a bipartite graph* $\mathsf{G} = (\mathcal{L} \cup \mathcal{R}, \mathsf{E})$*, has an* $\mathcal{L}$*-matching* $M \iff$ *for all* $L \subseteq \mathcal{L}$*, we have* $|L| \leq |N(L)|$.

Solution:

Maybe do only the first part of the proof in the discussion section, and sketch the second part.

Proof: **$\mathcal{L}$-matching $\implies$ Hall's condition.** One direction is easy, if there is an $\mathcal{L}$-matching M in G, then for any set L, we have that M contains $|L|$ edges in the matching M that covers the vertices of L. The endpoints of these edges are distinct, and they are all contained in $\mathcal{R}$. To this end, let

$$R = R(M, L) = \{ r \in \mathcal{R} \mid \ell r \in M \text{ and } \ell \in L \},$$

and observe that $|R| = |L|$. Clearly, $R \subseteq N(L)$. Implying that $|L| = |R| \leq |N(L)|$.

$\overline{\mathcal{L}\text{-matching}} \implies \overline{\text{Hall's condition}}$. Assume there is no $\mathcal{L}$-matching M in G. Then, consider a maximum matching M in G. there must be a free vertex $\ell \in L$ that is not in M. Let L be all the vertices on $\mathcal{L}$ reachable by an alternating path in G starting in L. Similarly, let R be the set of all vertices in $\mathcal{R}$ reachable by an alternating path starting at ℓ. Observe that:

(I) R can not contain any free vertex, as then there would be an alternating path π from a free vertex to a free vertex, which implies that $M \oplus \pi$ is a bigger matching. A contradiction to the assumption that M is maximum matching.

(II) L contains ℓ (duh).

(III) L contains no other free vertex. Indeed, all the vertices in L alternatingly reachable from ℓ, have a matching edge as their last edge (duh^2 [but really, all of math is a sequence of obvious observations]).

(IV) $|L| = |R| + 1$. Indeed, any vertex in R is attached to a unique vertex in $L - \ell$ by the matching M — otherwise it would be free, and that is not legal, because of ((I)).

(V) $N(L) = R$. Clearly, $R \subseteq N(L)$. As for the other direction, Consider any vertex $v \in L$. There is an alternating path from ℓ to v, denote it by π. The last edge vu in this path is a matching edge, and $u \in R$. All other edges vx, that are not in the matching, can be used to define a longer alternating path $\pi \mid vx$, implying that $x \in R$. Namely, $N(v) \subseteq R$, and thus $N(L) \subseteq R$.

We are done, as $|L| = |R| + 1 = |N(L)| + 1$. Namely, Hall's condition fails for L, as $|L| > |N(L)|$. ■

3 Partition a deck.

Consider a standard deck of cards – there are 13 ranks (1, . . . , 10, Princess, Queen and King. There are 4 suits: ♥, ♦, ♧, ♤ (thus 52 cards overall). Consider dividing the cards into piles of 4 cards, where no pile contains the same number twice. Show, that one can select exactly one card from each pile, such that overall we get all 13 possible values.

Solution:

Build the natural graph – there are 13 piles on the left, and 13 values no the right. Since this graph is 4-regular, it has a perfect matching (by Hall's theorem), and this matching is the desired way of picking the cards.

4 Consider a bipartite graph $\mathsf{G} = (\mathcal{L} \cup \mathcal{R}, \mathsf{E})$ that is k-regular (i.e., all vertices have the same degree k):

4.A. Prove that $|\mathcal{L}| = |\mathcal{R}|$.

Solution:

Observe that $|\mathsf{E}| = k|\mathcal{L}|$, and $|\mathsf{E}| = k|\mathcal{R}|$. We conclude that $|\mathcal{L}| = |\mathcal{R}|$.

4.B. Prove that there is a perfect matching in G.

Solution:

Indeed, for any set $L \subseteq \mathcal{L}$, consider its set of neighbors on the right side $R = N(L) \subseteq \mathcal{R}$. Let U be the set of edges between L and R in the graph. Observe that $|U| = k|L|$, and $|U| \leq k|R|$. We conclude that $|L| \leq |R|$, which is Hall's theorem condition. We conclude that this graph contains a perfect matching.

5 Given a k-regular bipartite graph, describe an algorithm that color the edges with k colors, such that no two edges with the same color share a vertex.

Solution:

In the ith iteration, the algorithm computes a maximum matching, M_i, removes its edges from G and repeats. Since the graph is $(k - i + 1)$-regular and bipartite, by the above, it has a perfect matching. This implies that M_i, for all i, covers all the vertices of G. Coloring all the edges of M_i by the ith color then implies the result.

6 Let R and B be two sets of n points in the plane. Consider the natural bipartite graph $\mathsf{G} = (R \cup B, \mathsf{E})$, where the length of an edge is the distance between the two points connected by this edge. Describe a polynomial time algorithm that computes a prefect matching M between R and B, that minimizes the longest edge in M.

Solution:

Let $Z = \{|pq| \mid p \in R, q \in B\}$ be the set of distances between the point. Using binary search, find the minimum distance r, such that the bipartite graph $(R \cup L, \mathsf{E}_{\leq r})$ contains a perfect matching (by computing a maximum matching in this graph. Here

$$\mathsf{E}_{\leq r} = \{pq \mid p \in R, q \in B, \|p - q\| \leq r\}.$$

Computing the set Z, and this set takes $O(n^2)$ time. Computing the maximum matching in such a graph takes $O(n^3)$ time. Overall, the running time of the algorithm is thus $O(n^3 \log n)$.