Describe recursive backtracking algorithms for the following problems. *Don't worry about running times.*

1 LONGEST INCREASING SUBSEQUENCE. Given an array $A[1..n]$ of integers, compute the length of a *longest increasing subsequence*. A sequence $B[1..\ell]$ is ***increasing*** if $B[i] > B[i-1]$ for every index $i \geq 2$.

For example, given the array

$$\langle 3, \underline{1}, \underline{4}, 1, \underline{5}, 9, 2, \underline{6}, 5, 3, 5, \underline{8}, \underline{9}, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$$

your algorithm should return the integer 6, because $\langle 1, 4, 5, 6, 8, 9 \rangle$ is a longest increasing subsequence (one of many).

Solution:

[#1 of ∞]

Add a sentinel value $A[0] = -\infty$. Let $LIS(i, j)$ denote the length of the longest increasing subsequence of $A[j..n]$ where every element is larger than $A[i]$. This function obeys the following recurrence:

$$\mathbf{LIS}(i,j) = \begin{cases} 0 & \text{if } j > n \\ \mathbf{LIS}(i, j+1) & \text{if } j \leq n \text{ and } A[i] \geq A[j] \\ \max\{\mathbf{LIS}(i, j+1), 1 + LIS(j, j+1)\} & \text{otherwise} \end{cases}$$

We need to compute $LIS(0, 1)$.

Solution:

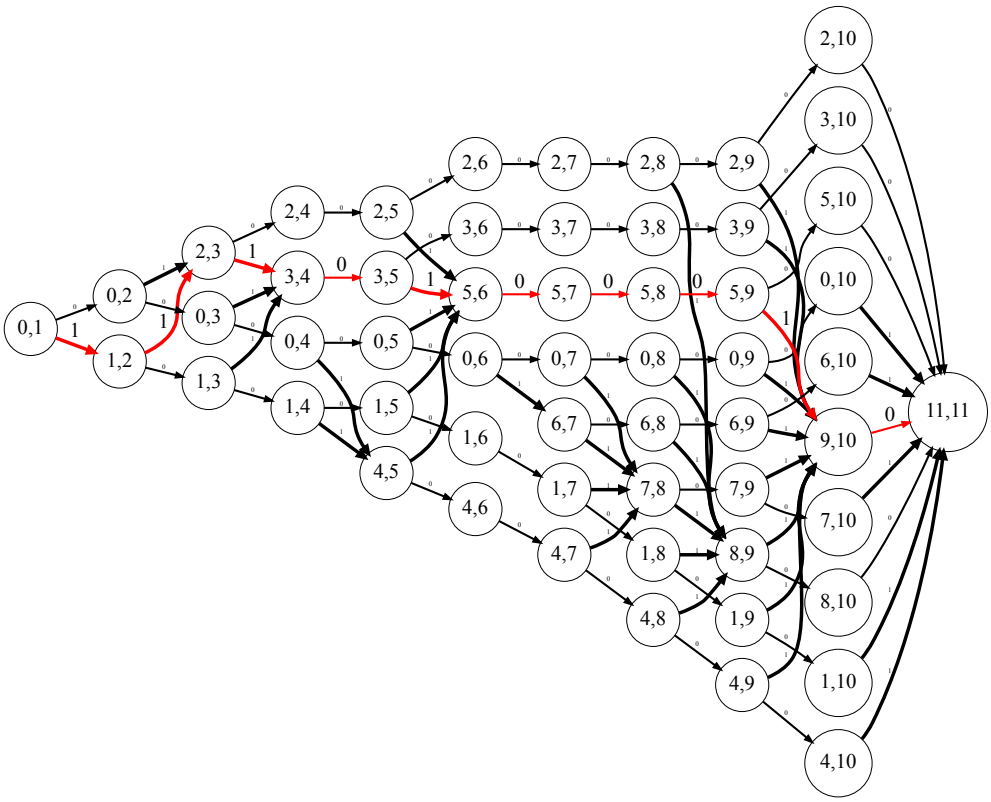
DP as a graph problem. All (or almost all) dynamic programming problems can be represented as a graph problem on a DAG. We show how to do it in this case – it is not very natural here – however, there are many cases where doing it using a graph yields much cleaner and nicer algorithms.

Details. (Building on the previous solution.) An alternative way to think about this problem, is as a configurations graph. Each distinct recursive call of **LIS**(i, j) is a distinct node (i, j). We add an edge between $(i, j) \to (i', j')$ if **LIS**(i, j) directly calls **LIS**(i', j'). We label each such edge by 0 or 1 depending if **LIS**adds one when considering the solution during the recursive call. In this specific case, this happens for an edge outgoing from (i, j) if $A[i] < A[j]$.

Thus, for the input:

$$A = [12, 60, 71, 17, 75, 6, 34, 63, 77, 37]$$

We get the graph:



(Here, for clarity, we replaced all the vertices of the form $(i, 11)$ by $(11, 11)$. We are now looking for the longest path (in the total sum of the labels of the edges along the path)in this DAG from $(0, 1)$ to $(11, 11)$. This path is (edges with no label have 0 label):

$$(0,1) \xrightarrow{1} (1,2) \xrightarrow{1} (2,3) \xrightarrow{1} (3,4) \Rightarrow (3,5) \xrightarrow{1} (5,6) \Rightarrow (5,7) \Rightarrow (5,8) \Rightarrow (5,9) \xrightarrow{1} (9,10) \Rightarrow (11,11),$$

This path has length 5. To read the solution from this path, we take the second number in each state that has an outgoing edge in this path with label 1. This, the solution is:

$$(0,\underline{1}) \xrightarrow{1} (1,\underline{2}) \xrightarrow{1} (2,\underline{3}) \xrightarrow{1} (3,4) \Rightarrow (3,\underline{5}) \xrightarrow{1} (5,6) \Rightarrow (5,7) \Rightarrow (5,8) \Rightarrow (5,\underline{9}) \xrightarrow{1} (9,10) \Rightarrow (11,11),$$

That is, the required longest increasing subsequence is $A[1], A[2], A[3], A[5], A[9]$. Since the input is $A = [12, 60, 71, 17, 75, 6, 34, 63, 77, 37]$, the result is thus the subsequence:

$$A = [\underline{12}, \underline{60}, \underline{71}, 17, \underline{75}, 6, 34, 63, \underline{77}, 37]$$

On the longest path. Computing the longest path in a graph is a NP-HARD problem (i.e., we believe it requires exponential time. However, on a DAG we can solve it in linear time in the size of the graph, as would see later in the class.

Solution:

[#2 of ∞] Add a sentinel value $A[n+1] = -\infty$. Let $LIS(i, j)$ denote the length of the longest increasing subsequence of $A[1 \,..\, j]$ where every element is smaller than $A[j]$. This function obeys the following recurrence:

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 1 \\ LIS(i-1,j) & \text{if } i \geq 1 \text{ and } A[i] \geq A[j] \\ \max\{LIS(i-1,j), 1 + LIS(i-1,i)\} & \text{otherwise} \end{cases}$$

We need to compute $LIS(n, n+1)$.

Solution: [#3 of ∞] Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i \,..\, n]$ that begins with $A[i]$. This function obeys the following recurrence:

$$LIS(i) = \begin{cases} 1 & \text{if } A[j] \leq A[i] \text{ for all } j > i \\ 1 + \max\{LIS(j)\}\ j > i \text{ and } A[j] > A[i] & \text{otherwise} \end{cases}$$

(The first case is actually redundant if we define $\max \varnothing = 0$.) We need to compute $\max_i LIS(i)$.

Solution: [#4 of ∞] Add a sentinel value $A[0] = -\infty$. Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i \,..\, n]$ that begins with $A[i]$. This function obeys the following recurrence:

$$LIS(i) = \begin{cases} 1 & \text{if } A[j] \leq A[i] \text{ for all } j > i \\ 1 + \max\{LIS(j)\}\ j > i \text{ and } A[j] > A[i] & \text{otherwise} \end{cases}$$

(The first case is actually redundant if we define $\max \varnothing = 0$.) We need to compute $LIS(0) - 1$; the -1 removes the sentinel $-\infty$ from the start of the subsequence.

Solution: [#5 of ∞] Add sentinel values $A[0] = -\infty$ and $A[n+1] = \infty$. Let $LIS(j)$ denote the length of the longest increasing subsequence of $A[1 \,..\, j]$ that ends with $A[j]$. This function obeys the following recurrence:

$$LIS(j) = \begin{cases} 1 & \text{if } j = 0 \\ 1 + \max\{LIS(i)\}\ i < j \text{ and } A[i] < A[j] & \text{otherwise} \end{cases}$$

We need to compute $LIS(n+1) - 2$; the -2 removes the sentinels $-\infty$ and ∞ from the subsequence.

2 LONGEST DECREASING SUBSEQUENCE. Given an array $A[1..n]$ of integers, compute the length of a *longest decreasing subsequence*. A sequence $B[1..\ell]$ is ***decreasing*** if $B[i] < B[i-1]$ for every index $i \geq 2$.

For example, given the array

$$\langle 3, 1, 4, 1, 5, \underline{9}, 2, \underline{6}, 5, 3, \underline{5}, 8, 9, 7, 9, 3, 2, 3, 8, \underline{4}, 6, \underline{2}, 7 \rangle$$

your algorithm should return the integer 5, because $\langle 9, 6, 5, 4, 2 \rangle$ is a longest decreasing subsequence (one of many).

Solution: [one of many] Add a sentinel value $A[0] = \infty$. Let $LDS(i, j)$ denote the length of the longest decreasing subsequence of $A[j \,..\, n]$ where every element is smaller than $A[i]$. This function obeys the following recurrence:

$$LDS(i,j) = \begin{cases} 0 & \text{if } j > n \\ LDS(i, j+1) & \text{if } j \le n \text{ and } A[i] \le A[j] \\ \max\{LDS(i, j+1), 1 + LIS(j, j+1)\} & \text{otherwise} \end{cases}$$

We need to compute $LDS(0, 1)$.

Solution: [clever] Multiply every element of A by -1, and then compute the length of the longest increasing subsequence using the algorithm from problem 1.

3 Longest alternating subsequence.

Given an array $A[1..n]$ of integers, compute the length of a *longest alternating subsequence*. A sequence $B[1..\ell]$ is ***alternating*** if $B[i] < B[i-1]$ for every even index $i \ge 2$, and $B[i] > B[i-1]$ for every odd index $i \ge 3$.

For example, given the array

$$\langle \underline{3}, \underline{1}, \underline{4}, \underline{1}, \underline{5}, 9, \underline{2}, \underline{6}, \underline{5}, 3, 5, \underline{8}, 9, \underline{7}, \underline{9}, \underline{3}, 2, 3, \underline{8}, \underline{4}, \underline{6}, \underline{2}, \underline{7} \rangle ,$$

your algorithm should return 17, because $\langle 3, 1, 4, 1, 5, 2, 6, 5, 8, 7, 9, 3, 8, 4, 6, 2, 7 \rangle$ is a longest alternating subsequence (one of many).

Solution: [one of many] We define two functions:

- Let $LAS^+(i, j)$ denote the length of the longest alternating subsequence of $A[j \,..\, n]$ whose first element (if any) is larger than $A[i]$ and whose second element (if any) is smaller than its first.
- Let $LAS^-(i, j)$ denote the length of the longest alternating subsequence of $A[j \,..\, n]$ whose first element (if any) is smaller than $A[i]$ and whose second element (if any) is larger than its first.

These two functions satisfy the following mutual recurrences:

$$LAS^+(i,j) = \begin{cases} 0 & \text{if } j > n \\ LAS^+(i, j+1) & \text{if } j \le n \text{ and } A[j] \le A[i] \\ \max\{LAS^+(i, j+1), 1 + LAS^-(j, j+1)\} & \text{otherwise} \end{cases}$$

$$LAS^-(i,j) = \begin{cases} 0 & \text{if } j > n \\ LAS^-(i, j+1) & \text{if } j \le n \text{ and } A[j] \ge A[i] \\ \max\{LAS^-(i, j+1), 1 + LAS^+(j, j+1)\} & \text{otherwise} \end{cases}$$

To simplify computation, we consider two different sentinel values $A[0]$. First we set $A[0] = -\infty$ and let $\ell^+ = LAS^+(0, 1)$. Then we set $A[0] = +\infty$ and let $\ell^- = LAS^-(0, 1)$. Finally, the length of the longest alternating subsequence of A is $\max\{\ell^+, \ell^-\}$.

Solution: [one of many] We define two functions:

- Let $LAS^+(i)$ denote the length of the longest alternating subsequence of $A[i \,..\, n]$ that starts with $A[i]$ and whose second element (if any) is larger than $A[i]$.
- Let $LAS^-(i)$ denote the length of the longest alternating subsequence of $A[i \,..\, n]$ that starts with $A[i]$ and whose second element (if any) is smaller than $A[i]$.

These two functions satisfy the following mutual recurrences:

$$LAS^+(i) = \begin{cases} 1 & \text{if } A[j] \le A[i] \text{ for all } j > i \\ 1 + \max\left\{LAS^-(j)\right\} j > i \text{ and } A[j] > A[i] & \text{otherwise} \end{cases}$$

$$LAS^-(i) = \begin{cases} 1 & \text{if } A[j] \ge A[i] \text{ for all } j > i \\ 1 + \max\left\{LAS^+(j)\right\} j > i \text{ and } A[j] < A[i] & \text{otherwise} \end{cases}$$

We need to compute $\max_i \max\left\{LAS^+(i), LAS^-(i)\right\}$.

To think about later:

1 Given an array $A[1..n]$ of integers, compute the length of a longest ***convex*** subsequence of A.

Solution: Let $LCS(i, j)$ denote the length of the longest convex subsequence of $A[i \,..\, n]$ whose first two elements are $A[i]$ and $A[j]$. This function obeys the following recurrence:

$$LCS(i, j) = 1 + \max\{LCS(j, k)\} \; j < k \le n \text{ and } A[i] + A[k] > 2A[j]$$

Here we define $\max \varnothing = 0$; this gives us a working base case. The length of the longest convex subsequence is $\max_{1 \le i < j \le n} LCS(i, j)$.

Solution: [with sentinels] Assume without loss of generality that $A[i] \ge 0$ for all i. (Otherwise, we can add $|m|$ to each $A[i]$, where m is the smallest element of $A[1 \,..\, n]$.) Add two sentinel values $A[0] = 2M + 1$ and $A[-1] = 4M + 3$, where M is the largest element of $A[1 \,..\, n]$.
Let $LCS(i, j)$ denote the length of the longest convex subsequence of $A[i \,..\, n]$ whose first two elements are $A[i]$ and $A[j]$. This function obeys the following recurrence:

$$LCS(i, j) = 1 + \max\{LCS(j, k)\} \; j < k \le n \text{ and } A[i] + A[k] > 2A[j]$$

Here we define $\max \varnothing = 0$; this gives us a working base case.
Finally, we claim that the length of the longest convex subsequence of $A[1 \,..\, n]$ is $LCS(-1, 0) - 2$.

Proof: First, consider any convex subsequence S of $A[1 \,..\, n]$, and suppose its first element is $A[i]$. Then we have $A[-1] - 2A[0] + A[i] = 4M + 3 - 2(2M + 1) + A[i] = A[i] + 1 > 0$, which implies that $A[-1] \cdot A[0] \cdot S$ is a convex subsequence of $A[-1 \,..\, n]$. So the longest convex subsequence of $A[1 \,..\, n]$ has length at most $LCS(-1, 0) - 2$.
On the other hand, removing $A[-1]$ and $A[0]$ from any convex subsequence of $A[-1 \,..\, n]$ laves a convex subsequence of $A[1 \,..\, n]$. So the longest subsequence of $A[1 \,..\, n]$ has length at least $LCS(-1, 0) - 2$.■

2 Given an array $A[1..n]$, compute the length of a longest ***palindrome*** subsequence of A.

Solution: [naive] Let $LPS(i, j)$ denote the length of the longest palindrome subsequence of $A[i \,..\, j]$. This function obeys the following recurrence:

$$LPS(i,j) = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ \max\left\{ \begin{array}{l} LPS(i+1,j) \\ LPS(i,j-1) \end{array} \right\} & \text{if } i < j \text{ and } A[i] \neq A[j] \\ \max\left\{ \begin{array}{l} 2 + LPS(i+1,j-1) \\ \quad LPS(i+1,j) \\ \quad LPS(i,j-1) \end{array} \right\} & \text{otherwise} \end{cases}$$

We need to compute $LPS(1, n)$.

Solution: [with greedy optimization] Let $LPS(i, j)$ denote the length of the longest palindrome subsequence of $A[i \,..\, j]$. Before stating a recurrence for this function, we make the following useful observation.[a]

Claim 0.1. *If $i < j$ and $A[i] = A[j]$, then* $\text{LPS}(i, j) = 2 + \text{LPS}(i + 1, j - 1)$.

Proof: Suppose $i < j$ and $A[i] = A[j]$. Fix an arbitrary longest palindrome subsequence S of $A[i \,..\, j]$. There are four cases to consider.

- If S uses neither $A[i]$ nor $A[j]$, then $A[i] \bullet S \bullet A[j]$ is a palindrome subsequence of $A[i \,..\, j]$ that is longer than S, which is impossible.
- Suppose S uses $A[i]$ but not $A[j]$. Let $A[k]$ be the last element of S. If $k = i$, then $A[i] \bullet A[j]$ is a palindrome subsequence of $A[i \,..\, j]$ that is longer than S, which is impossible. Otherwise, replacing $A[k]$ with $A[j]$ gives us a palindrome subsequence of $A[i \,..\, j]$ with the same length as S that uses both $A[i]$ and $A[j]$.
- Suppose S uses $A[j]$ but not $A[i]$. Let $A[h]$ be the first element of S. If $h = j$, then $A[i] \bullet A[j]$ is a palindrome subsequence of $A[i \,..\, j]$ that is longer than S, which is impossible. Otherwise, replacing $A[h]$ with $A[i]$ gives us a palindrome subsequence of $A[i \,..\, j]$ with the same length as S that uses both $A[i]$ and $A[j]$.
- Finally, S might include both $A[i]$ and $A[j]$.

In all cases, we find either a contradiction or a longest palindrome subsequence of $A[i \,..\, j]$ that uses both $A[i]$ and $A[j]$. ■

Claim 1 implies that the function LPS satisfies the following recurrence:

$$LPS(i,j) = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ \max\big\{LPS(i+1,j),\ LPS(i,j-1)\big\} & \text{if } i < j \text{ and } A[i] \neq A[j] \\ 2 + LPS(i+1,j-1) & \text{otherwise} \end{cases}$$

We need to compute $LPS(1, n)$.

[a] And yes, optimizations like this require a proof of correctness, both in homework and on exams. Premature optimization is the root of all evil.