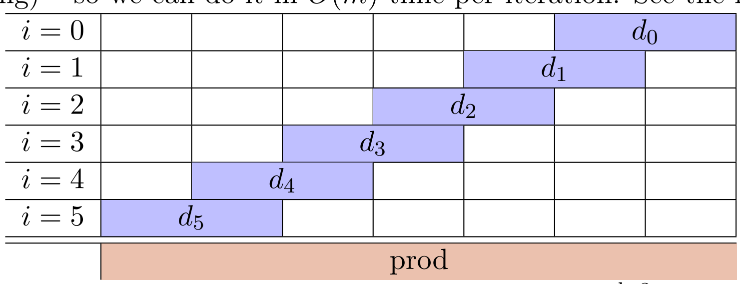
In lecture, Alex described an algorithm of Karatsuba that multiplies two n-digit integers using $O(n^{\lg 3})$ single-digit additions, subtractions, and multiplications. In this lab we'll look at some extensions and applications of this algorithm.

1 Describe an algorithm to compute the product of an n-digit number and an m-digit number, where $m < n$, in $O(m^{\lg 3 - 1} n)$ time.

Solution: Split the larger number into $\lceil n/m \rceil$ chunks, each with m digits. Multiply the smaller number by each chunk in $O(m^{\lg 3})$ time using Karatsuba's algorithm, and then add the resulting partial products with appropriate shifts.

```
SkewMultiply(x[0 .. m − 1], y[0 .. n − 1]):
  prod ← 0
  offset ← 0
  for i ← 0 to ⌈n/m⌉ − 1
      c_i ← y[i · m . . . (i + 1) · m − 1]
      d_i ← Multiply(x, c_i)
      prod ← prod + d_i · 10^(i·m)        (*)
  return prod
```

Each call to **Multiply** requires $O(m^{\lg 3})$ time, and all other work within a single iteration of the main loop requires $O(m)$ time. To see why (*) indeed takes $O(m)$ time, observe that in the ith iteration, we add a number of $2m$ digits to the current sum prod, but we do it in the top part of the current sum (because of the shifting) – so we can do it in $O(m)$ time per iteration. See the following illustration.



Thus, the overall running time of the algorithm is $O(1) + \lceil n/m \rceil\, O(m^{\lg 3}) = O(m^{\lg 3 - 1} n)$ as required. This is the standard method for multiplying a large integer by a single "digit" integer ***written in base*** $\boldsymbol{10^m}$, but with each single-"digit" multiplication implemented using Karatsuba's algorithm.

2 Describe an algorithm to compute the decimal representation of 2^n in $O(n^{\lg 3})$ time. (The standard algorithm that computes one digit at a time requires $\Theta(n^2)$ time.)

Solution:

We compute 2^n via repeated squaring, implementing the following recurrence:

$$2^n = \begin{cases} 1 & \text{if } n = 0 \\ (2^{n/2})^2 & \text{if } n > 0 \text{ is even} \\ 2 \cdot (2^{\lfloor n/2 \rfloor})^2 & \text{if } n \text{ is odd} \end{cases}$$

We use Karatsuba's algorithm to implement decimal multiplication for each square.

```
TwoToThe(n):
  if n = 0
     return 1
  m ← ⌊n/2⌋
  z ← TwoToThe(m)          // recurse!
  z ← Multiply(z, z)       // Karatsuba
  if n is odd
     z ← Add(z, z)
  return z
```

The running time of this algorithm satisfies the recurrence

$$T(n) = T(\lfloor n/2 \rfloor) + O(n^{\lg 3}).$$

We can safely ignore the floor in the recursive argument. The recursion tree for this algorithm is just a path; the work done at recursion depth i is $O((n/2^i)^{\lg 3}) = O(n^{\lg 3}/3^i)$. Thus, the levels sums form a descending geometric series, which is dominated by the work at level 0, so the total running time is at most $O(n^{\lg 3})$.

3 Describe a divide-and-conquer algorithm to compute the decimal representation of an arbitrary n-bit binary number in $O(n^{\lg 3})$ time. (**Hint:** Let $x = a \cdot 2^{n/2} + b$. Watch out for an extra log factor in the running time.)

Solution:

Following the hint, we break the input x into two smaller numbers $x = a \cdot 2^{n/2} + b$; recursively convert a and b into decimal; convert $2^{n/2}$ into decimal using the solution to problem 2; multiply a and $2^{n/2}$ using Karatsuba's algorithm; and finally add the product to b to get the final result.

```
Decimal(x[0 .. n − 1]):
  if n < 100
     use brute force
  m ← ⌈n/2⌉
  a ← x[m .. n − 1]
  b ← x[0 .. m − 1]
  return Add(Multiply(Decimal(a), TwoToThe(m)), Decimal(b))
```

The running time of this algorithm satisfies the recurrence

$$T(n) = 2T(n/2) + O(n^{\lg 3}).$$

The $O(n^{\lg 3})$ term includes the running times of both **Multiply** and **TwoToThe** (as well as the final linear-time addition).

The recursion tree for this algorithm is a binary tree, with 2^i nodes at recursion depth i. Each recursive call at depth i converts an $n/2^i$-bit binary number to decimal. The non-recursive work at the corresponding node of the recursion tree, of depth i, is

$$W_i = O((n/2^i)^{\lg 3}) = O(n^{\lg 3}/3^i).$$

There are 2^i nodes in the recursion tree of depth i. As such, the total work at all the nodes of the recursion tree of depth i is

$$L_i = 2^i \cdot W_i = 2^i \cdot O(n^{\lg 3}/3^i) = O\big(n^{\lg 3}/(3/2)^i\big).O\big(n^{\lg 3}(2/3)^i\big).$$

The total running time of the algorithm is bounded by

$$\sum_{i=0}^{\infty} L_i = \sum_{i=0}^{\infty} O\big(n^{\lg 3}(2/3)^i\big) = O\big(n^{\lg 3}\big) \sum_{i=0}^{\infty}(2/3)^i = O\big(n^{\lg 3}\big),$$

since $\sum_{i=0}^{\infty}(2/3)^i = O(1)$ (specifically 3).

Notice that if we had converted $2^{n/2}$ to decimal *recursively* instead of calling **TwoToThe**, the recurrence would have been $T(n) = 3T(n/2) + O(n^{\lg 3})$. Every level of this recursion tree has the same sum, so the overall running time would be $O(n^{\lg 3}\log n)$.

Think about later:

4 Suppose we can multiply two n-digit numbers in $O(M(n))$ time. Describe an algorithm to compute the decimal representation of an arbitrary n-bit binary number in $O(M(n)\log n)$ time.

Solution:

We modify the solutions of problems 2 and 3 to use the faster multiplication algorithm instead of Karatsuba's algorithm. Let $T_2(n)$ and $T_3(n)$ denote the running times of **TwoToThe** and **Decimal**, respectively. We need to solve the recurrences

$$T_2(n) = T_2(n/2) + O(M(n)) \quad \text{and} \quad T_3(n) = 2T_3(n/2) + T_2(n) + O(M(n)).$$

But how can we do that when we don't know $M(n)$?

For the moment, suppose $M(n) = O(n^c)$ for some constant $c > 0$. Since any algorithm to multiply two n-digit numbers must *read* all n digits, we have $M(n) = \Omega(n)$, and therefore $c \geq 1$. On the other hand, the grade-school lattice algorithm implies $M(n) = O(n^2)$, so we can safely assume $c \leq 2$. With this assumption, the recursion tree method implies

$$T_2(n) = T_2(n/2) + O(n^c) \qquad \Longrightarrow T_2(n) = O(n^c)$$

$$T_3(n) = 2T_3(n/2) + O(n^c) \qquad \Longrightarrow T_3(n) = \begin{cases} O(n\log n) & \text{if } c = 1, \\ O(n^c) & \text{if } c > 1. \end{cases}$$

So in this case, we have $T_3(n) = O(M(n)\log n)$ as required.

In reality, $M(n)$ may not be a simple polynomial, but we can effectively ***ignore*** any sub-polynomial noise using the following trick. Suppose we can write $M(n) = n^c \cdot \mu(n)$ for some constant c and some arbitrary non-decreasing function $\mu(n)$.[1]

To solve the recurrence $T_2(n) = T_2(n/2) + O(M(n))$, we define a new function $\tilde{T}_2(n) = T_2(n)/\mu(n)$. Then we have

$$\tilde{T}_2(n) \;=\; \frac{T_2(n/2)}{\mu(n)} + \frac{O(M(n))}{\mu(n)} \;\leq\; \frac{T_2(n/2)}{\mu(n/2)} + \frac{O(M(n))}{\mu(n)} \;=\; \tilde{T}_2(n/2) + O(n^c).$$

Here we used the inequality $\mu(n) \geq \mu(n/2)$; this the only fact about μ that we actually need. The recursion tree method implies $\tilde{T}_2(n) \leq O(n^c)$, and therefore $T_2(n) \leq O(n^c)\cdot\mu(n) = O(M(n))$.

Similarly, to solve the recurrence $T_3(n) = 2T_3(n/2) + O(M(n))$, we define $\tilde{T}_3(n) = T_3(n)/\mu(n)$, which gives us the recurrence $\tilde{T}_3(n) \leq 2\tilde{T}_3(n/2) + O(n^c)$. The recursion tree method implies

$$\tilde{T}_3(n) \leq \begin{cases} O(n\log n) & \text{if } c = 1, \\ O(n^c) & \text{if } c > 1. \end{cases}$$

In both cases, we have $\tilde{T}_3(n) = O(n^c\log n)$, which implies that $T_3(n) = O(M(n)\log n)$.