Prove that each of the following languages is ***not*** regular.

1 $\{0^{2^n} \mid n \geq 0\}$.

Solution:

Let $F = L = \{0^{2^n} \mid n \geq 0\}$.

Let x and y be arbitrary elements of F.

Then $x = 0^{2^i}$ and $y = 0^{2^j}$ for some non-negative integers x and y.

Let $z = 0^{2^i}$.

Then $xz = 0^{2^i}0^{2^i} = 0^{2^{i+1}} \in L$.

And $yz = 0^{2^j}0^{2^i} = 0^{2^i+2^j} \notin L$, because $i \neq j$

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

Solution:

For any non-negative integers $i \neq j$, the strings 0^{2^i} and 0^{2^j} are distinguished by the suffix 0^{2^i}, because $0^{2^i}0^{2^i} = 0^{2^{i+1}} \in L$ but $0^{2^j}0^{2^i} = 0^{2^{i+j}} \notin L$. Thus L itself is an infinite fooling set for L.

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2 $\{0^{2n}1^n \mid n \geq 0\}$

Solution:

Let F be the language 0^*.

Let x and y be arbitrary strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i1^i$.

Then $xz = 0^{2i}1^i \in L$.

And $yz = 0^{i+j}1^i \notin L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

Solution:

For all non-negative integers $i \neq j$, the strings 0^i and 0^j are distinguished by the suffix 0^i1^i, because $0^{2i}1^i \in L$ but $0^{i+j}1^i \notin L$. Thus, the language 0^* is an infinite fooling set for L.

Solution:

For all non-negative integers $i \neq j$, the strings 0^{2i} and 0^{2j} are distinguished by the suffix 1^i, because $0^{2i}1^i \in L$ but $0^{2j}1^i \notin L$. Thus, the language $(00)^*$ is an infinite fooling set for L.

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3 $\{0^m1^n \mid m \neq 2n\}$

Solution:

Let F be the language 0^*.

Let x and y be arbitrary strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i1^i$.

Then $xz = 0^{2i}1^i \notin L$.

And $yz = 0^{i+j}1^i \in L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

Solution:

For all non-negative integers $i \neq j$, the strings 0^{2i} and 0^{2j} are distinguished by the suffix 1^i, because $0^{2i}1^i \notin L$ but $0^{2j}1^i \in L$. Thus, the language $(00)^*$ is an infinite fooling set for L.

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4 Strings over $\{0, 1\}$ where the number of 0s is exactly twice the number of 1s.

Solution:

Let F be the language 0^*.

Let x and y be arbitrary strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i1^i$.

Then $xz = 0^{2i}1^i \in L$.

And $yz = 0^{i+j}1^i \notin L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

Solution:

For all non-negative integers $i \neq j$, the strings 0^{2i} and 0^{2j} are distinguished by the suffix 1^i, because $0^{2i}1^i \in L$ but $0^{2j}1^i \notin L$. Thus, the language $(00)^*$ is an infinite fooling set for L.

Solution:

If L were regular, then the language

$$\big((0+1)^* \setminus L\big) \cap 0^*1^* \;=\; \{0^m1^n \mid m \neq 2n\}$$

would also be regular, because regular languages are closed under complement and intersection. But we just proved that $\{0^m1^n \mid m \neq 2n\}$ is not regular in problem 3. *[Yes, this proof would be worth full credit, either in homework or on an exam.]*

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5 Strings of properly nested parentheses (), brackets [], and braces {}. For example, the string ([]){} is in this language, but the string ([ )] is not, because the left and right delimiters don't match.

Solution:

Let F be the language $(^*$.

Let x and y be arbitrary strings in F.

Then $x = (^i$ and $y = (^j$ for some non-negative integers $i \neq j$.

Let $z =)^i$.

Then $xz = (^i)^i \in L$.

And $yz = (^j)^i \notin L$, because $i \neq j$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

Solution:

For any non-negative integers $i \neq j$, the strings $(^i$ and $(^j$ are distinguished by the suffix $)^i$, because $(^i)^i \in L$ but $(^i)^j \notin L$. Thus, the language $(^*$ is an infinite fooling set.

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6 Strings of the form $w_1\#w_2\#\cdots\#w_n$ for some $n \geq 2$, where each substring w_i is a string in $\{0,1\}^*$, and some pair of substrings w_i and w_j are equal.

Solution:

Let F be the language 0^*.

Let x and y be arbitrary strings in F.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = \#0^i$.

Then $xz = 0^i\#0^i \in L$.

And $yz = 0^j\#0^i \notin L$, because $i \neq j$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular.

Solution:

For any non-negative integers $i \neq j$, the strings 0^i and 0^j are distinguished by the suffix $\#0^i$, because $0^i\#0^i \in L$ but $0^j\#0^i \notin L$. Thus, the language 0^* is an infinite fooling set.

Extra problems

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7 $\{0^{n^2} \mid n \geq 0\}$

Solution:

Let x and y be distinct arbitrary strings in L.

Without loss of generality, $x = 0^{i^2}$ and $y = 0^{j^2}$ for some $j > i \geq 0$.

Let $z = 0^{2i+1}$.

Then $xz = 0^{i^2}0^{2i+1} = 0^{i^2+2i+1} = 0^{(i+1)^2} \in L$

On the other hand, $yz = 0^{j^2}0^{2i+1} = 0^{j^2+2i+1} \notin L$, because $j^2 < j^2 + 2i + 1 < j^2 + 2j + 1 = (j+1)^2$.

Thus, z distinguishes x and y.

We conclude that L is an infinite fooling set for itself (i.e., L), so L cannot be, will not be, and is not regular.

Solution:

Let x and y be distinct arbitrary strings in 0^*.

Without loss of generality, $x = 0^i$ and $y = 0^j$ for some $i > j \geq 0$.

Let $z = 0^{i^2+i+1}$.

Then $xz = 0^{i^2+2i+1} = 0^{(i+1)^2} \in L$.

On the other hand, $yz = 0^{i^2+i+j+1} \notin L$, because $i^2 < i^2 + i + j + 1 < (i+1)^2$.

Thus, z distinguishes x and y.

We conclude that 0^* is an infinite fooling set for L, so L cannot be regular.

Solution:

Let x and y be distinct arbitrary strings in 0000^*.

Without loss of generality, $x = 0^i$ and $y = 0^j$ for some $i > j \geq 3$.

Let $z = 0^{i^2-i}$.

Then $xz = 0^{i^2} \in L$.

On the other hand, $yz = 0^{i^2-i+j} \notin L$, because

$$(i-1)^2 \;=\; i^2 - 2i + 1 \;<\; i^2 - i \;<\; i^2 - i + j \;<\; i^2.$$

(The first inequalities requires $i \geq 2$, and the second $j \geq 1$.)

Thus, z distinguishes x and y.

We conclude that 0000^* is an infinite fooling set for L, so L cannot be regular.

8 $\{w \in (0+1)^* \mid w \text{ is the binary representation of a perfect square}\}$

Solution:

We design our fooling set around numbers of the form $(2^k+1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k1 \in L$, for any integer $k \geq 2$. The argument is somewhat simpler if we further restrict k to be even.

Let $F = 1(00)^*1$, and let x and y be arbitrary strings in F.

Then $x = 10^{2i-2}1$ and $y = 10^{2j-2}1$, for some positive integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let $z = 0^{2i}1$.

Then $xz = 10^{2i-2}10^{2i}1$ is the binary representation of $2^{4i} + 2^{2i+1} + 1 = (2^{2i}+1)^2$, and therefore $xz \in L$.

On the other hand, $yz = 10^{2j-2}10^{2i}1$ is the binary representation of $2^{2i+2j} + 2^{2i+1} + 1$. Simple algebra gives us the inequalities

$$\begin{aligned}
(2^{i+j})^2 &= 2^{2i+2j} \\
&< \boldsymbol{2^{2i+2j} + 2^{2i+1} + 1} \\
&< 2^{2(i+j)} + 2^{i+j+1} + 1 \\
&= (2^{i+j}+1)^2.
\end{aligned}$$

So $2^{2i+2j} + 2^{2i+1} + 1$ lies between two consecutive perfect squares, and thus is not a perfect square, which implies that $yz \notin L$.

We conclude that F is a fooling set for L. Because F is infinite, L cannot be regular.