

Designing DFAs via product construction and designing NFAs.

- 1 DFA for all strings in which the number of 0s is even and the number of 1s is *not* divisible by 3.

Solution:

We use a standard product construction of two DFAs, one accepting strings with an even number of 0s, and the other accepting strings where the number of 1s is not a multiple of 3.

The product DFA has six states, each labeled with a pair of integers, one indicating the number 0s read modulo 2, the other indicating the number of 1s read modulo 3.

$$Q := \{0, 1\} \times \{0, 1, 2\}$$

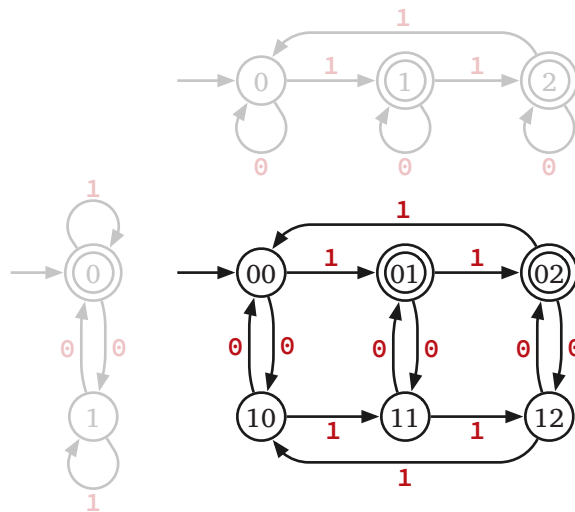
$$s := (0, 0)$$

$$A := \{(0, 1), (0, 2)\}$$

$$\delta((q, r), 0) := (q + 1 \bmod 2, r)$$

$$\delta((q, r), 1) := (q, r + 1 \bmod 3)$$

In this case, the product DFA is simple enough that we can just draw it out in full. I have drawn the two factor DFAs (in gray) to the left and above for reference.



2 DFA for all strings that are **both** the binary representation of an integer divisible by 3 **and** the ternary (base-3) representation of an integer divisible by 4.

For example, the string **1100** is an element of this language, because it represents $2^3 + 2^2 = 12$ in binary and $3^3 + 3^2 = 36$ in ternary.

Solution:

Again, we use a standard product construction of two DFAs, one accepting binary strings divisible by 3, the other accepting ternary strings divisible by 4. The product DFA has twelve states, each labeled with a pair of integers: The binary value read so far modulo 3, and the ternary value read so far modulo 4.

$$Q := \{0, 1, 2\} \times \{0, 1, 2, 3\}$$

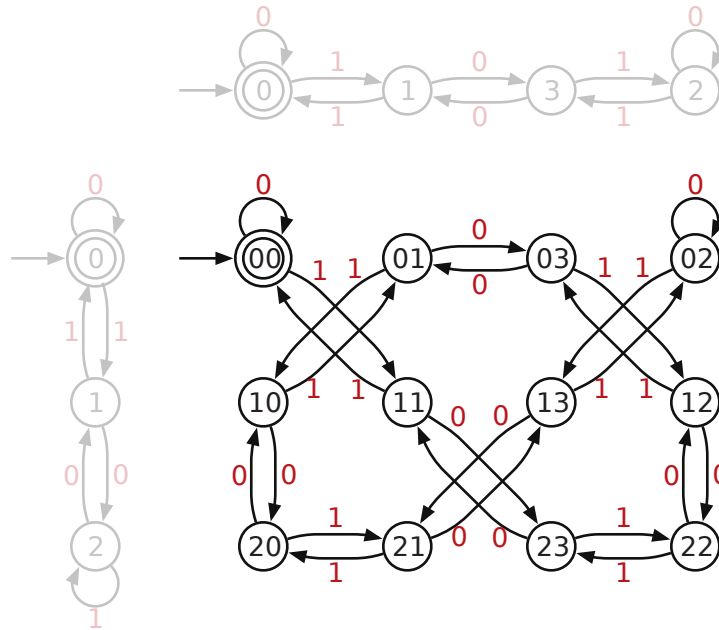
$$s := (0, 0)$$

$$A := \{(0, 0)\}$$

$$\delta((q, r), 0) := (2q \bmod 3, \quad 3r \bmod 4)$$

$$\delta((q, r), 1) := (2q + 1 \bmod 3, \quad 3r + 1 \bmod 4)$$

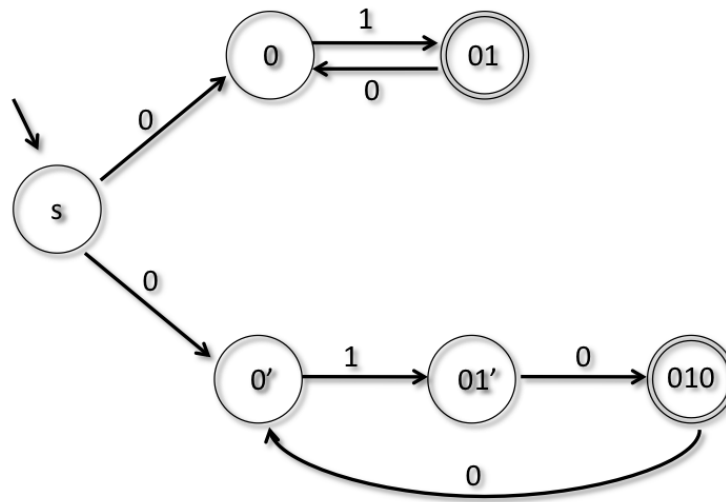
For reference, here is a drawing of the DFA, with the two factor DFAs (in gray) to the left and above; we would not expect you to draw this, especially on exams. More importantly we would expect you **not** to draw this, **especially** on exams. The states of the factor DFA that maintains ternary-value-mod-4 are deliberately “out of order” to simplify the drawing.



3 Design an NFA for the language $(01)^+ + (010)^+$.

Solution:

The NFA is shown in the figure below.



Note that we've separated the two cases of either repeated 01 , or repeated 010 . Why would the NFA with states labeled 0 and $0'$ merged be incorrect?

4 DFA for all strings w such that $\binom{|w|}{2} \bmod 6 = 4$. (**Hint:** Maintain both $\binom{|x|}{2} \bmod 6$ and $|x| \bmod 6$.)

Solution:

Our DFA has 36 states, each labeled with a pair of integers representing $\binom{|x|}{2} \bmod 6$ and $|x| \bmod 6$, where x is the prefix of the input read so far.

$$Q := \{0, 1, 2, 3, 4, 5\} \times \{0, 1, 2, 3, 4, 5\}$$

$$s := \{(0, 0)\}$$

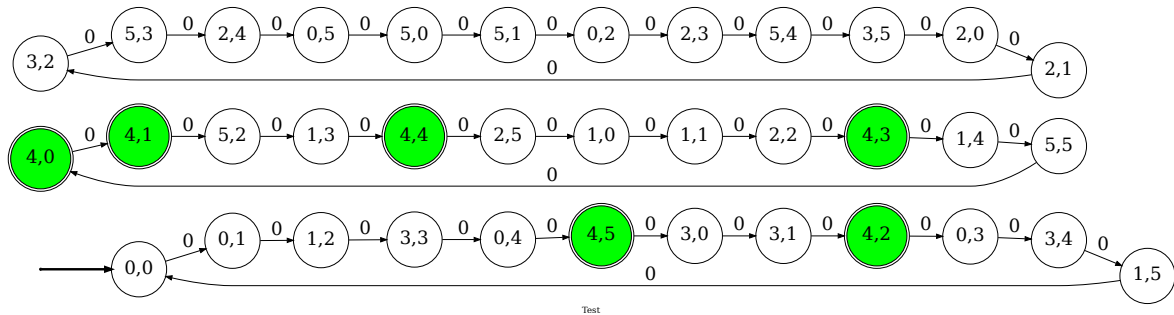
$$A := \{(4, r) \mid r \in \{0, 1, 2, 3, 4, 5\}\}$$

$$\delta((q, r), 0) := (q + r \bmod 6, r + 1 \bmod 6)$$

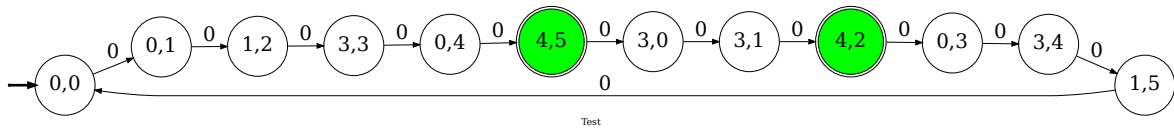
$$\delta((q, r), 1) := (q + r \bmod 6, r + 1 \bmod 6)$$

The transition function exploits the identity $\binom{n+1}{2} = \frac{(n+1)n}{2} = n + \frac{(n-1)n}{2} = \binom{n}{2} + n$.

The resulting DFA (we draw the transition using only 0 for clarity) is



But of course, there is no need to keep the portion of the DFA that is not reachable from the start state. This yields the smaller DFA:



The above automata shows that this language is all strings that their length is either 5 or 8 modulo 12.

5 (Hard.) All strings w such that $F_{\#(10,w)} \bmod 10 = 4$, where $\#(10,w)$ denotes the number of times 10 appears as a substring of w , and F_n is the n th Fibonacci number:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

Solution:

Our DFA has 200 states, each labeled with three values:

- $F_k \bmod 10$, where k is the number of times we have seen the substring 10.
- $F_{k+1} \bmod 10$, where k is the number of times we have seen the substring 10.
- The last symbol read (or 0 if we have read nothing yet)

Here is the formal description:

$$Q := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \times \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \times \{0, 1\}$$

$$s := \{(0, 1, 0)\}$$

$$A := \{(4, r, a) \mid r \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \text{ and } a \in \{0, 1\}\}$$

$$\delta((q, r, 0), 0) := (q, r, 0)$$

$$\delta((q, r, 1), 0) := (r, q + r \bmod 10, 0)$$

$$\delta((q, r, 0), 1) := (q, r, 1)$$

$$\delta((q, r, 1), 1) := (q, r, 1)$$

The transition function exploits the recursive definition $F_{k+1} = F_k + F_{k-1}$.

Solution:

The Fibonacci numbers modulo 10 define a repeating sequence with period 60. So this language can be accepted by a DFA with “only” 120 states.

Extra problems [no solutions would be provided]

6 Let $L = \{w \in \{a, b\}^* \mid \text{an } a \text{ appears in some position } i \text{ of } w, \text{ and a } b \text{ appears in position } i + 2\}$.

- 6.A. Create an NFA N for L with at most four states.
- 6.B. Using the “power-set” construction, create a DFA M from N . Rather than writing down the sixteen states and trying to fill in the transitions, build the states as needed, because you won’t end up with unreachable or otherwise superfluous states.
- 6.C. Now directly design a DFA M' for L with only five states, and explain the relationship between M and M' .