1 You are given a set of T of m toys, and there are n children. The ith child has a set of toys $T_i \subseteq T$ that they are willing to play with. Decide if there is a way of giving a toy to each child, so that they are all happy (i.e., playing with a toy they like). Assuming the input size is $O(nm)$ (why?), what is the running time of your algorithm solving this problem?

2 Prove Hall's theorem:

Theorem 0.1 (Hall's theorem). *For a bipartite graph* $\mathsf{G} = (\mathcal{L} \cup \mathcal{R}, \mathsf{E})$*, has an* $\mathcal{L}$*-matching* $M \iff$ *for all* $L \subseteq \mathcal{L}$*, we have* $|L| \leq |N(L)|$.

3 Partition a deck.

Consider a standard deck of cards – there are 13 ranks $(1, \ldots, 10$, Princess, Queen and King. There are 4 suits: ♥, ♦, ♧, ♤ (thus 52 cards overall). Consider dividing the cards into piles of 4 cards, where no pile contains the same number twice. Show, that can select exactly one card from each pile, such that overall we get all 13 possible values.

4 Consider a bipartite graph $\mathsf{G} = (\mathcal{L} \cup \mathcal{R}, \mathsf{E})$ that is k-regular (i.e., all vertices have the same degree k):

4.A. Prove that $|\mathcal{L}| = |\mathcal{R}|$.

4.B. Prove that there is a perfect matching in G.

5 Given a k-regular bipartite graph, describe an algorithm that color the edges with k colors, such that no two edges with the same color share a vertex.

6 Let R and B be two sets of n points in the plane. Consider the natural bipartite graph $\mathsf{G} = (R \cup B, \mathsf{E})$, where the length of an edge is the distance between the two points connected by this edge. Describe a polynomial time algorithm that computes a prefect matching M between R and B, that minimizes the longest edge in M.