HW 11: Extra problems

Instructors: Har-Peled

CS/ECE 374A: Intro. Algorithms & Models of Computation, Fall 2024 Version: **1.0**

1 (100 PTS.) Walk with me

1.A. (50 PTS.) We are given a *directed* graph G with n vertices and m edges ($m \geq n$), where each vertex v has a height $h(v)$. The *cost* of traversing an edge (u, v) is $c(u, v) = |h(v) - h(u)|$. The cost of a walk in G is the sum of the costs of edges in the walk. Prove that finding the minimum cost walk that visits all the vertices om G is NP-HARD. (In a walk, vertices and edges may be repeated, and the start and end vertices may be different.)

1.B. (50 PTS.) We are given a directed graph G with n vertices and m edges ($m \geq n$), where each edge e has a set of colors $C(e) \subseteq \{1, ..., k\}$. Prove that deciding whether there exists a walk that uses all k colors (i.e. the union of the sets of colors of the edges of walk covers all colors.) is NP-HARD. (Hint: Reduce from Set Cover.)

2 (100 PTS.) Things are hard.

2.A. (20 PTS.) Suppose we have n prisoners $P_1, \ldots, P_n$ that we want to place in some disconnected blocks of a prison. Each prisoner is assigned to one block, and he/she will not be able to access other blocks. However, some prisoners are bitter enemies (going all the way back to kindergarten) and cannot be placed in the same block. Given integers n and k and a list of enemies for each of the n prisoners, we want to determine whether k blocks are sufficient to house all the prisoners? Prove that this problem is NP-HARD. You can safely assume that every block has unbounded capacity.

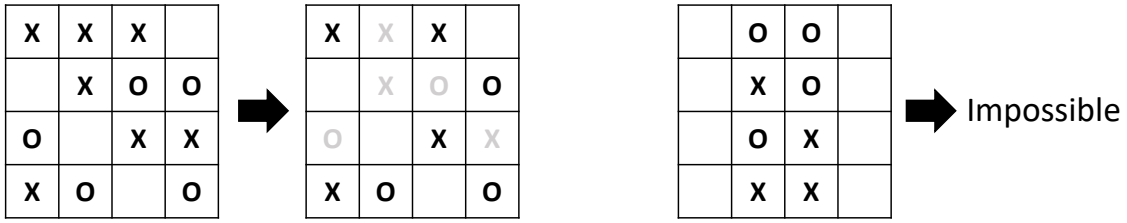
2.B. (40 PTS.) Let G be an arbitrary directed weighted graph with n vertices and m edges such that no edge weight is zero (weights can be positive or negative). Prove that finding a *zero-length* (i.e., zero weight) Hamiltonian cycle in G is NP-COMPLETE.

2.C. (40 PTS.) Consider the following **XO** puzzle. You are given an $n \times m$ grid of squares where each square has an **X**, an **O** or is empty. Your goal is to erase some of the **X**s and **O**s so that

(i) every row contains at least one **X** or one **O**, and

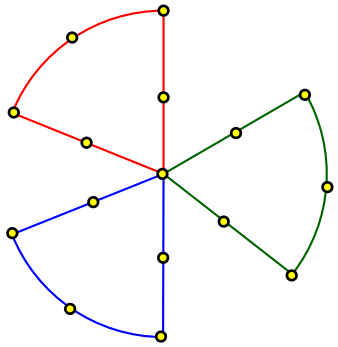
(ii) no column contains both **X** and **O**.

For some input grids, it is impossible to solve the puzzle. The figure below shows two examples: a grid that is solvable and a grid that is impossible to solve. Prove that, given a grid, it is NP-HARD to determine whether the puzzle is solvable. (Hint: Reduce from 3SAT.)



3 (100 PTS.) Fan, fan, fan.

An undirected graph a ***3-blade-fan*** if it consists of three cycles C_1, C_2, and C_3 of k nodes each and they all share exactly one node. Hence, the graph has $3k - 2$ nodes. The figure below shows a ***3-blade-fan*** of 16 nodes.



Given an undirected graph G with n vertices and m edges and an integer k, the FAN problem asks whether or not there exists a subgraph of G which is a ***3-blade-fan***. Prove that FAN is NP-COMPLETE.

4 A subset S of vertices in an undirected graph G is called ***almost independent*** if at most 374 edges in G have both endpoints in S. Prove that finding the size of the largest almost-independent set of vertices in a given undirected graph is NP-hard.

5 A subset S of vertices in an undirected graph G is called ***triangle-free*** if, for every triple of vertices $u, v, w \in S$, at least one of the three edges uv, uw, vw is *absent* from G. Prove that finding the size of the largest triangle-free subset of vertices in a given undirected graph is NP-hard.

6 Charon needs to ferry n recently deceased people across the river Acheron into Hades. Certain pairs of these people are sworn enemies, who cannot be together on either side of the river unless Charon is also present. (If two enemies are left alone, one will steal the obol from the other's mouth, leaving them to wander the banks of the Acheron as a ghost for all eternity. Let's just say this is a Very Bad Thing.) The ferry can hold at most k passengers at a time, including Charon, and only Charon can pilot the ferry.

Prove that it is NP-hard to decide whether Charon can ferry all n people across the Acheron unharmed.[1] The input for Charon's problem consists of the integers k and n and an n-vertex graph G describing the pairs of enemies. The output is either TRUE or FALSE.

This problem is a generalization of the following extremely well-known puzzle, whose first known appearance is in the treatise *Propositiones ad Acuendos Juvenes* [*Problems to Sharpen the Young*] by the 8th-century English scholar Alcuin of York.[2]

> XVIII. PROPOSITIO DE HOMINE ET CAPRA ET LVPO.
>
> Homo quidam debebat ultra fluuium transferre lupum, capram, et fasciculum cauli. Et non potuit aliam nauem inuenire, nisi quae duos tantum ex ipsis ferre ualebat. Praeceptum itaque ei fuerat, ut omnia haec ultra illaesa omnino transferret. Dicat, qui potest, quomodo eis illaesis transire potuit?

[1] Aside from being, you know, dead.

[2] At least, we *think* that's who wrote it; the evidence for his authorship is rather circumstantial, although we do know from his correspondence with Charlemagne that he sent the emperor some "simple arithmetical problems for fun". Most scholars believe that even if Alcuin is the actual author of the *Propositiones*, he didn't come up with the problems himself, but just collected his problems from other sources. Some things never change.

Solutio. Simili namque tenore ducerem prius capram et dimitterem foris lupum et caulum. Tum deinde uenirem, lupumque transferrem: lupoque foris misso capram naui receptam ultra reducerem; capramque foris missam caulum transueherem ultra; atque iterum remigassem, capramque assumptam ultra duxissem. Sicque faciendo facta erit remigatio salubris, absque uoragine lacerationis.

In case your classical Latin is rusty, here is an English translation:

XVIII. The Problem of the Man, the Goat, and the Wolf.

A man needed to transfer a wolf, a goat, and a bundle of cabbage across a river. However, he found that his boat could only bear the weight of two [objects at a time, including the man]. And he had to get everything across unharmed. Tell me if you can: How they were able to cross unharmed?

Solution. In a similar fashion [as an earlier problem], I would first take the goat across and leave the wolf and cabbage on the opposite bank. Then I would take the wolf across; leaving the wolf on shore, I would retrieve the goat and bring it back again. Then I would leave the goat and take the cabbage across. And then I would row across again and get the goat. In this way the crossing would go well, without any threat of slaughter.

Please do not write your solution to problem 3 in classical Latin.

7 Consider an instance of the Satisfiability Problem, specified by clauses $C_1, \ldots, C_k$ over a set of Boolean variables $x_1, \ldots, x_n$. We say that the instance is *monotone* if each term in each clause consists of a nonnegated variable; that is each term is equal to x_i, for some i, rather than $\bar{x}_i$. Monotone instance of Satisfiability are very easy to solve: They are always satisfiable, by setting each variable equal to 1.

For example, suppose we have the three clauses

$$(x_1 \vee x_2), (x_1 \vee x_3), (x_2 \vee x_3)$$

This is monotone, and indeed the assignment that sets all three variables to 1 satisfies all the clauses. But we can observe that this is not the only satisfying assignment; we could also have set x_1 and x_2 to 1 and x_3 to 0. Indeed, for any monotone instance, it is natural to ask how few variables we need to set to 1 in order to satisfy it.

Given a monotone instance of Satisfiability, together with a number k, the problem of *Monotone Satisfiability with Few True Variables* asks: Is there a satisfying assignment for the instance in which at most k variables are set to 1? Prove that this problem is NP-Complete. *Hint:* Reduce from Vertex Cover.

8 Given an undirected graph $G = (V, E)$, a partition of V into $V_1, V_2, \ldots, V_k$ is said to be a clique cover of size k if each V_i is a clique in G. Prove that the problem of deciding whether G has a clique cover of size at most k is NP-Complete. *Hint:* Consider the complement of G.

9 Given an undirected graph $G = (V, E)$ a *matching* in G is a set of edges $M \subseteq E$ such that no two edges in M share a node. A matching M is *perfect* if $2|M| = |V|$, in other words if every node is incident to some edge of M. PerfectMatching is the following decision problem: does a given graph G have a perfect matching? Describe a polynomial-time reduction from PerfectMatching to SAT. Does this problem that PerfectMatching is a difficult problem?

10 A balloon is a directed graph on an even number of nodes, say $2n$, in which n of the nodes form a directed cycle and the remaining n vertices are connected in a "tail" that consists of a directed path joined to one of the nodes in the cycle. See figure below for a balloon with 8 nodes.

Given a directed graph G and an integer k, the BALLOON problem asks whether or not there exists a subgraph which is a baloon that contains $2k$ nodes. Prove that BALLOON is NP-Complete.

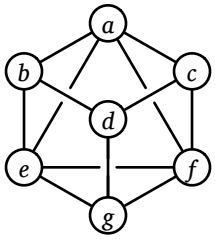
11 Consider the following problem. You are managing a communication network, modeled by a directed graph $G = (V, E)$. There are c *users* who are interested in making use of this network. User i (for each $i = 1, 2, \ldots, c$) issues a *request* to reserve a specific path P_i in G on which to transmit data.

You are interested in accepting as many of these path requests as possible, subject to the following restriction: if you accept both P_i and P_j, then P_i and P_j can not share any modes.

Thus the *Path Selection Problem* asks: Given a directed graph $G = (V, E)$, a set of requests $P_1, \ldots, P_c$-each of which must be a path in G- and a number k, is it possible to select at least k of the paths so that no two of the selected paths share any nodes?

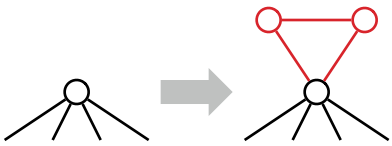
Prove that the Path Selection is NP-Complete.

12 A *double-Hamiltonian tour* in an undirected graph G is a closed walk that visits every vertex in G exactly twice. Prove that it is NP-hard to decide whether a given graph G has a double-Hamiltonian tour.



This graph contains the double-Hamiltonian tour $a \to b \to d \to g \to e \to b \to d \to c \to f \to a \to c \to f \to g \to e \to a$.

Solution: We prove the problem is NP-hard with a reduction from the standard Hamiltonian cycle problem. Let G be an arbitrary undirected graph. We construct a new graph H by attaching a small gadget to every vertex of G. Specifically, for each vertex v, we add two vertices $v^\sharp$ and $v^\flat$, along with three edges $vv^\flat$, $vv^\sharp$, and $v^\flat v^\sharp$.



A vertex in G, and the corresponding vertex gadget in H.

I claim that G has a Hamiltonian cycle if and only if H has a double-Hamiltonian tour.

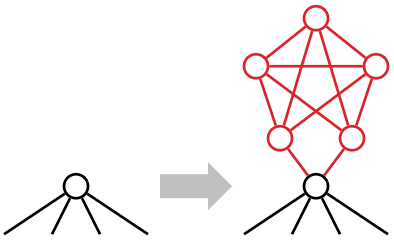
$\Longrightarrow$ Suppose G has a Hamiltonian cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$. We can construct a double-Hamiltonian tour of H by replacing each vertex v_i with the following walk:

$$\cdots \to v_i \to v_i^\flat \to v_i^\sharp \to v_i^\flat \to v_i^\sharp \to v_i \to \cdots$$

$\Longleftarrow$ Conversely, suppose H has a double-Hamiltonian tour D. Consider any vertex v in the original graph G; the tour D must visit v exactly twice. Those two visits split D into two closed walks, each of which visits v exactly once. Any walk from $v^\flat$ or $v^\sharp$ to any other vertex in H must pass through v. Thus, one of the two closed walks visits only the vertices v, $v^\flat$, and $v^\sharp$. Thus, if we simply remove the vertices in $H \setminus G$ from D, we obtain a closed walk in G that visits every vertex in G once.

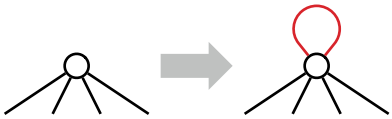
Given any graph G, we can clearly construct the corresponding graph H in polynomial time.

With more effort, we can construct a graph H that contains a double-Hamiltonian tour ***that traverses each edge of H at most once*** if and only if G contains a Hamiltonian cycle. For each vertex v in G we attach a more complex gadget containing five vertices and eleven edges, as shown on the next page.



A vertex in G, and the corresponding modified vertex gadget in H.

Solution: Bad and incorrect solution!!!
We attempt to prove the problem is NP-hard with a reduction from the Hamiltonian cycle problem. Let G be an arbitrary undirected graph. We construct a new graph H by attaching a self-loop every vertex of G. Given any graph G, we can clearly construct the corresponding graph H in polynomial time.

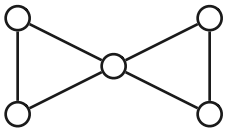


An incorrect vertex gadget.

Suppose G has a Hamiltonian cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$. We can construct a double-Hamiltonian tour of H by alternating between edges of the Hamiltonian cycle and self-loops:

$$v_1 \to v_1 \to v_2 \to v_2 \to v_3 \to \cdots \to v_n \to v_n \to v_1.$$

On the other hand, if H has a double-Hamiltonian tour, we *cannot* conclude that G has a Hamiltonian cycle, because we cannot guarantee that a double-Hamiltonian tour in H uses *any* self-loops. The graph G shown below is a counterexample; it has a double-Hamiltonian tour (even before adding self-loops) but no Hamiltonian cycle.



This graph has a double-Hamiltonian tour.

Rubric:[for all polynomial-time reductions] 10 points =

+ 3 points for the reduction itself
  - For an NP-hardness proof, the reduction must be from a known NP-hard problem. You can use any of the NP-hard problems listed in the lecture notes (except the one you are trying to prove NP-hard, of course).
+ 3 points for the "if" proof of correctness
+ 3 points for the "only if" proof of correctness
+ 1 point for writing "polynomial time"

- An incorrect polynomial-time reduction that still satisfies half of the correctness proof is worth at most 4/10.
- A reduction in the wrong direction is worth 0/10.

13 (100 PTS.) My friend, departing time is pending (Fall 2022).

The following question is long, but not very hard, and is intended to make sure you understand the following problems, and the basic concepts needed for proving NP-Completeness.

All graphs in the following have n vertices and m edges.

For each of the following problems, you are given an instance of the problem of size n. Imagine that the answer to this given instance is "yes", and that you need to convince somebody that indeed the answer to the given instance is `yes`. To this end, describe:

(I) An algorithm for solving the given instance (not necessarily efficient). What is the running time of your algorithm?
(II) The format of the proof that the instance is correct.
(III) A bound on the length of the proof (its have to be of polynomial length in the input size).
(IV) An efficient algorithm (as fast as possible [it has to be polynomial time]) for verifying, given the instance and the proof, that indeed the given instance is indeed `yes`. What is the running time of your algorithm?

(EXAMPLE)

Shortest Path

Instance: A weighted undirected graph G, vertices s and t and a threshold w.
Question: Is there a path between s and t in G of length at most w?

Solution:

(I) **Algorithm**: We seen in class the Dijkstra algorithm for solving the shortest path problem in $O(n \log n + m) = O(n^2)$ time. Given the shortest path, we can just compare its price to w, and return yes/no accordingly.
(II) **Certificate**: A "proof" in this case would be a path π in G (i.e., a sequence of at most n vertices) connecting s to t, such that its total weight is at most w.
(III) **Certificate length**: The proof here is a list of $O(n)$ vertices, and can be encoded as a list of $O(n)$ integers. As such, its length is $O(n)$.
(IV) **Verification algorithm**: The verification algorithm for the given solution/proof, would verify that all the edges in the path are indeed in the graph, the path starts at s and ends at t, and that the total weight of the edges of the path is at most w. The proof has length $O(n)$ in this case, and the verification algorithm runs in $O(n^2)$ time, if we assume the graph is given to us using adjacency lists representation.

13.A. (20 PTS.)

Friendly Set

Instance: An undirected graph G, integer k
Question: Is there a friendly set in G of size k?

A set $X \subseteq \mathsf{V}(\mathsf{G})$ is ***friendly*** if every vertex has at least two vertices in X that are its neighbors. Formally, for a vertex $u \in \mathsf{V}(\mathsf{G})$, let $\Gamma(u) = \{v \mid uv \in \mathsf{E}(\mathsf{G})\}$ be its set of ***neighbors***. The set X is friendly if for all $v \in \mathsf{V}(\mathsf{G})$, we have that $|\Gamma(v) \cap X| \geq 2$.

13.B. (20 PTS.)

Max Triangle Free

Instance: An undirected graph G with n vertices and m edges, a parameter k.
Question: Is there a subset S of k edges in the graph, such that no three edges of S form a triangle?

A ***triangle*** is a cycle of length 3.

13.C. (20 PTS.)

Wiggle to target

Instance: S: Set of positive integers. t: An integer number (target).
Question: Are there disjoint subsets $X, Y, Z \subseteq S$ such that $\sum_{x \in X} x - \sum_{y \in Y} y + 2\sum_{z \in Z} z = t$?

13.D. (20 PTS.)

NotMuchOverlap

Instance: X a set of n elements, $\mathcal{F} = \{f_i \subseteq X \mid i = 1, \ldots, m\}$, and a parameter α.
Question: Is there a subset $S \subseteq \mathcal{F}$ of α sets, such that no pair of sets $f, g \in S$ share more than one element.

13.E. (20 PTS.)

SET DISJOINT PACKING

Instance: $(\mathsf{U}, \mathcal{F}, k)$:
U: A set of n elements
$\mathcal{F}$: A family of m subsets of U, s.t. $\bigcup_{X \in \mathcal{F}} X = \mathsf{U}$.
k: A positive integer.
Question: Are there k pairwise-disjoint sets $S_1, \ldots, S_k \in \mathcal{F}$?

Formally, the sets $S_1, \ldots, S_k$ are ***pairwise-disjoint*** if for all $i \neq j$, we have that $S_i \cap S_j = \emptyset$.

14 (100 PTS.) Set picking (Fall 2022).

SET PICKING

Instance: A set $U = \{1, \ldots, m\}$, and sets $\mathcal{F} = \{f_1, \ldots, f_n, g_1, \ldots, g_n \subseteq U\}$.
Question: Is there a good selection set $\mathcal{X} \subseteq \mathcal{F}$?

A ***good selection*** is a set $\mathcal{X} \subseteq \mathcal{F}$, such that $|\mathcal{X} \cap \{f_i, g_i\}| = 1$, for all i, and furthermore $\cup\mathcal{X} = \cup_{h \in \mathcal{X}} h = U$.

14.A. (20 PTS.) Prove that SET PICKING is in NP.

14.B. (40 PTS.) Show a polynomial time reduction from SET PICKING to SAT. (You need to prove your reduction is correct.)

14.C. (40 PTS.) Show a polynomial time reduction from SAT to SET PICKING. (You need to prove your reduction is correct.)