Midterm 2 review

Lecture 22

Part I

Recursion: Divide and Conquer

Recursion types

1. **Divide and Conquer**: Problem reduced to multiple **independent** sub-problems.

   Examples: Binary search, Merge sort, quick sort, multiplication, median selection.

   Each sub-problem is a fraction smaller.

Binary Search

1. Discard half every time

Binary Search

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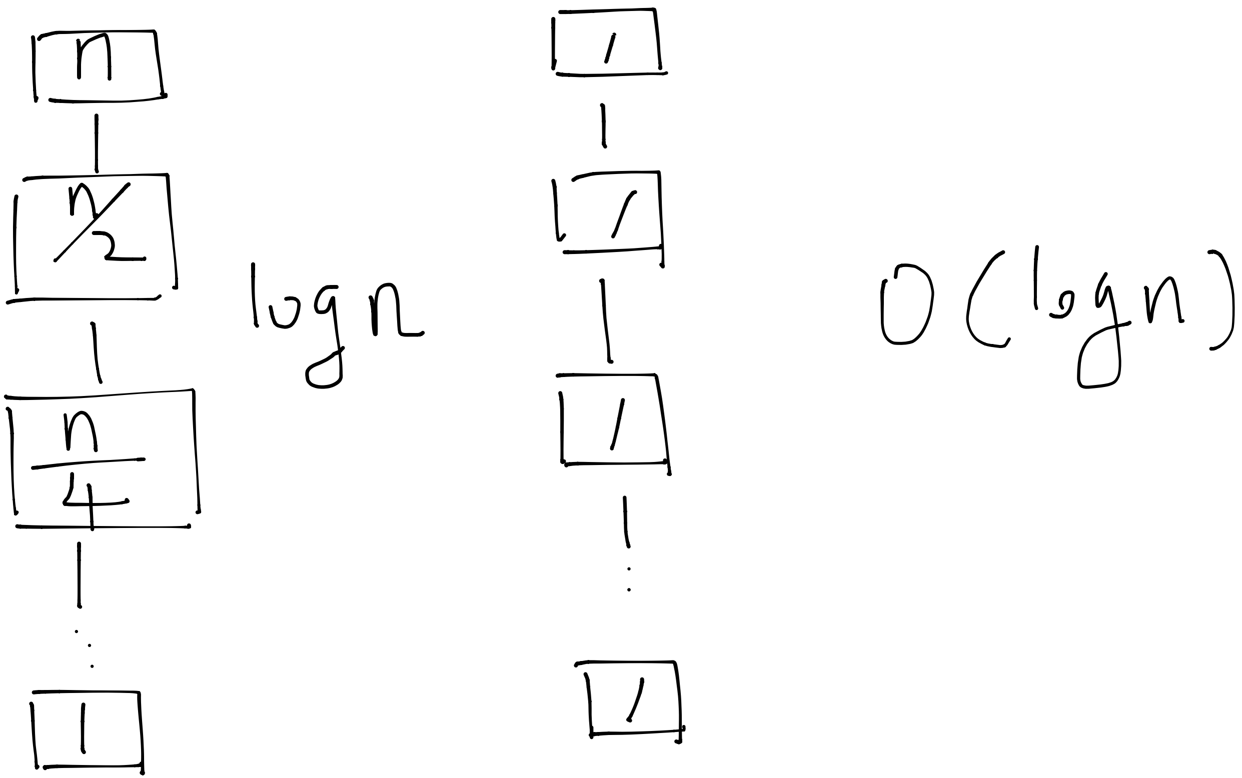
2. Recurrence tree



Binary Search

1. Discard half every time
2. Recurrence tree
3. Which condition to check?

Binary Search

Suppose you are given two sorted arrays $A[1 .. n]$ and $B[1 .. n]$ containing distinct integers. Describe a fast algorithm to find the median (meaning the nth smallest element) of the union $A \cup B$. For example, given the input

$$A[1 .. 8] = [0, 1, 6, 9, 12, 13, 18, 20]$$

$$B[1 .. 8] = [2, 4, 5, 8, 17, 19, 21, 23]$$

your algorithm should return the integer **9**.

6

$A_1 < 4$ of elements

$A_1 <$ median

$\leq A_3$

$A_1 \quad B_1$

$A_2 < B_2$

$A_3 \quad B_3$

$A_1 \leq A_2 < B_2 \leq B_3$

Binary Search

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$$B[1..8] = [2, 4, 5, 8, 17, 19, 21, 23]$$

your algorithm should return the integer 9.

Compare the two medians.

Binary Search

$\underline{\text{MEDIAN}(A[1..n], B[1..n]):}$

```
if n < 10^100
    use brute force
else if A[n/2] > B[n/2]
    return MEDIAN(A[1..n/2], B[n/2+1..n])
else
    return MEDIAN(A[n/2+1..n], B[1..n/2])
```

Binary Search

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  else
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```

Because we discard the same number of elements from each array, the median of the remaining subarrays is the median of the original $A \cup B$.

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1. Divide into two halves. Together takes $O(n)$ time.

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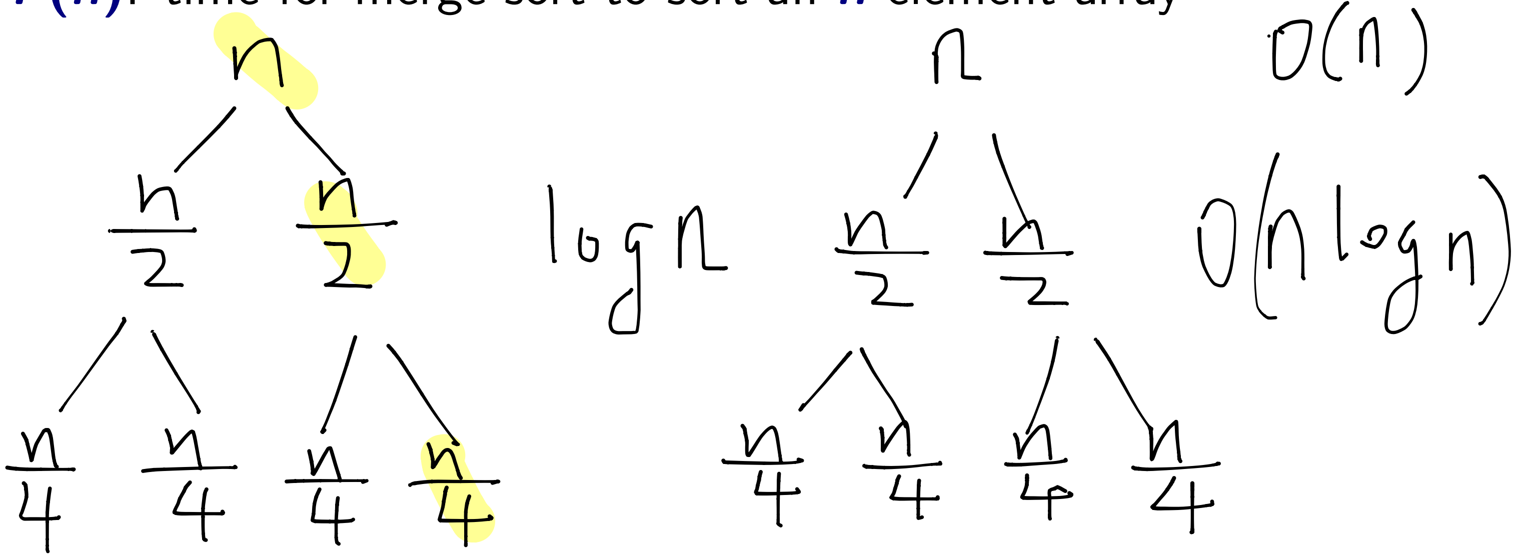

Sorting

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$\boldsymbol{T(n)}$: time for merge sort to sort an $\boldsymbol{n}$ element array

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

Karatsuba's Algorithm

$$xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R)$$
$$= 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

Gauss trick: $x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$

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Time Analysis

Running time is given by

$$T(n) = 3T(n/2) + O(n) \qquad T(1) = O(1)$$

which means

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Time Analysis

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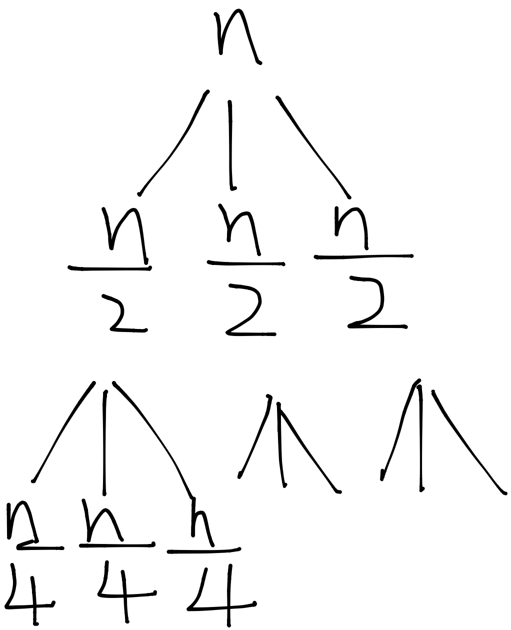
$$T(n) = 3T(n/2) + O(n) \qquad T(1) = O(1)$$

which means $T(n) = O(n^{\log_2 3}) = O(n^{1.585})$

Recursion tree analysis



log n

$$\sum_{i=0}^{\log n} \frac{3^i}{2^i} n$$

↑ increasing

$$O\left(\left(\frac{3}{2}\right)^{\log n} n\right)$$

$$\frac{3^{\log n}}{n} n = 3^{\log n} = 2^{(\log 3)(\log n)}$$
$$= n^{\log 3}$$

Selecting in Unsorted Lists

1. One-armed Quick-sort

Selecting in Unsorted Lists

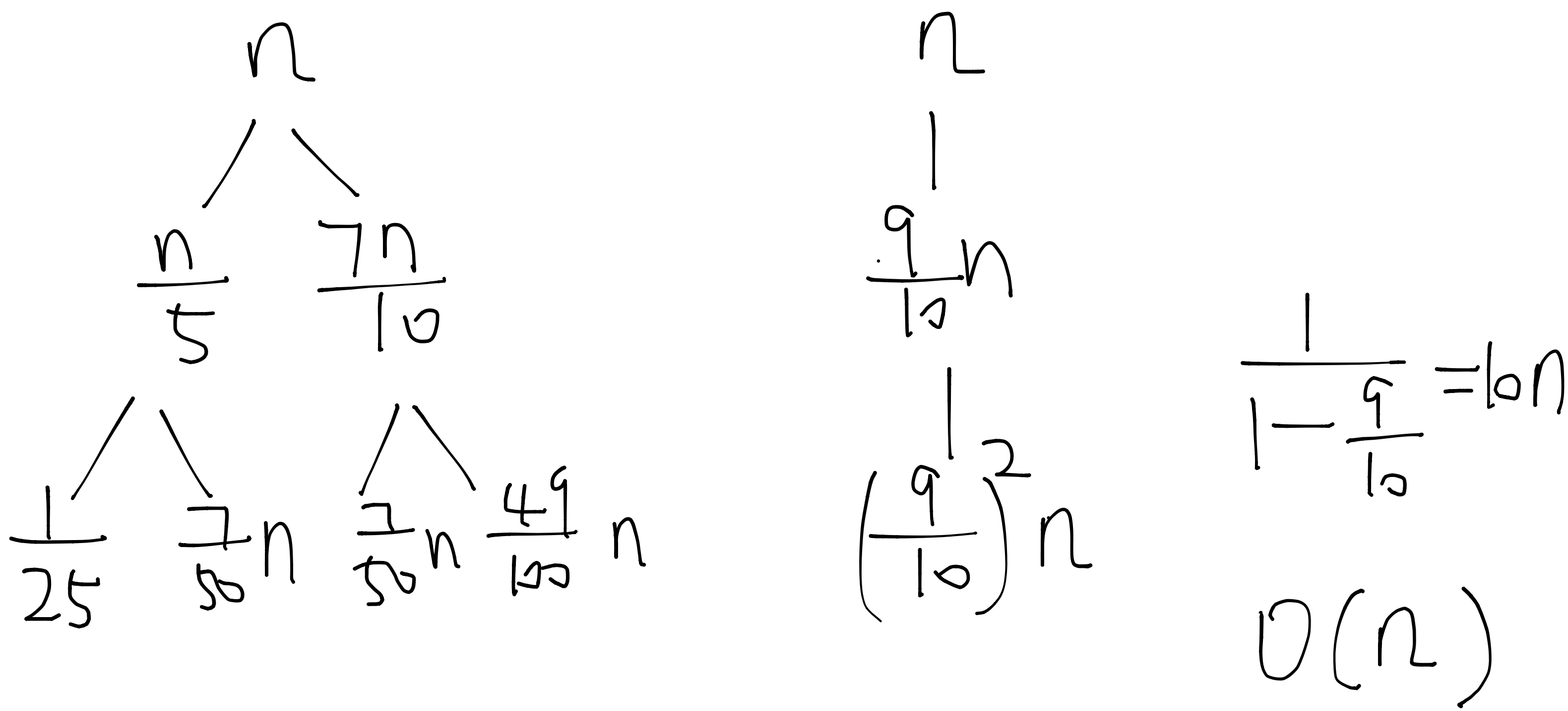
1. One-armed Quick-sort
2. With a good pivot (median of the medians)

$$T(n) \leq T(\lceil n/5 \rceil) + T(\lceil 7n/10 \rceil) + O(n)$$

and

$$T(n) = O(1) \qquad n < 10$$

Recursion tree analysis



Part II

Dynamic programming

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3. **Dynamic Programming**: Smart recursion with memoization

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- Changes the problem into a sequence of decision problems
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Max-Weight Independent Set in Trees: Two possibilities: Include the root or not

How to design DP algorithms

1. Find a “smart” recursion (The hard part)
   1. Formulate the sub-problem
   2. so that the number of distinct subproblems is small; polynomial in the original problem size.

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   2. so that the number of distinct subproblems is small; polynomial in the original problem size.

2. Memoization
   1. Identify distinct subproblems
   2. Choose a memoization data structure
   3. Identify dependencies and find a good evaluation order
   4. An iterative algorithm replacing recursive calls with array lookups
   5. Further optimize space

Which data structure?

- Text segmentation, suffix, 1-D array
- Longest increasing subsequence, suffix+index, 2-D array
- Edit distance, two prefixes, 2-D array
- Max-Weight Independent Set in Trees, tree

Part III

Graphs

Path and cycle

A path is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k-1$. The length of the path is $k-1$ (the number of edges in the path) and the path is from v_1 to v_k.
Note: a single vertex u is a path of length 0.

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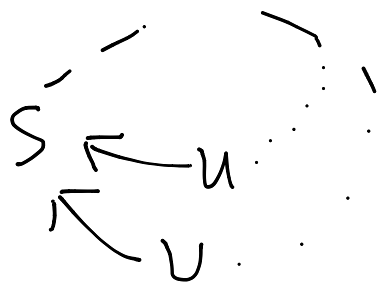
A cycle is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k-1$ and ~~$\{v_1, v_k\}$~~ $\in E$. Single vertex not a cycle according to this definition.

$\{v_k, v_1\}$



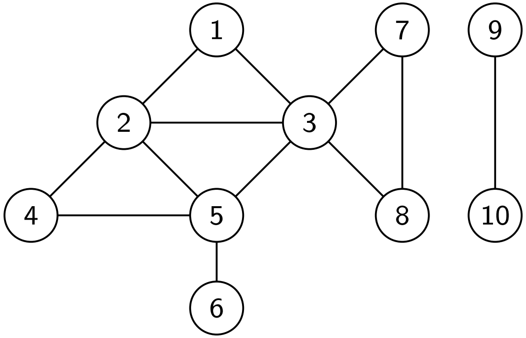
$d(s \to u) + w(u \to s)$

$d(s \to v) + w(v \to s)$

min

Connectivity on Undirected Graphs

Given a graph $G = (V, E)$:



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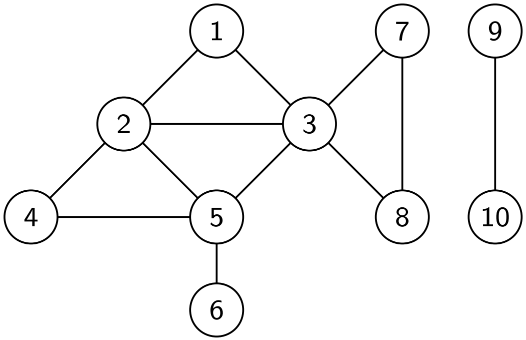
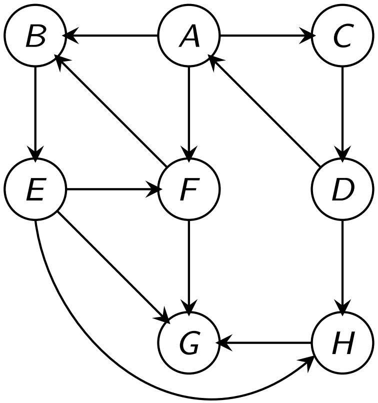


A vertex u is connected to v if there is a path from u to v.

The connected component of u, $\text{con}(u)$, is the set of all vertices connected to u.

Directed Connectivity

Given a graph $G = (V, E)$:



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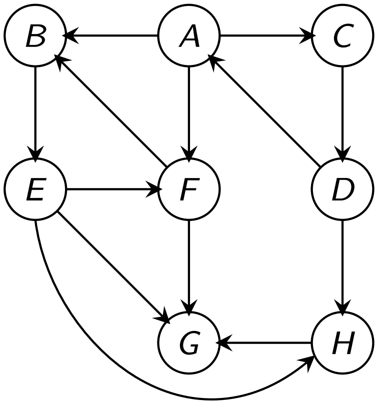

A vertex u can reach v if there is a path from u to v.

Let $\mathbf{rch}(u)$ be the set of all vertices reachable from u.

Asymmetricity: D can reach B but B cannot reach D

Connectivity and Strong Connected Components

Definition

Given a directed graph G, u is strongly connected to v if u can reach v *and* v can reach u. In other words $v \in \mathrm{rch}(u)$ and $u \in \mathrm{rch}(v)$.

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C is an equivalence relation, that is reflexive, symmetric and transitive.

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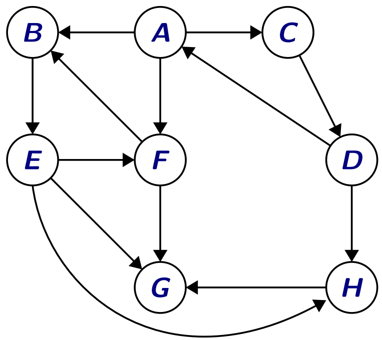
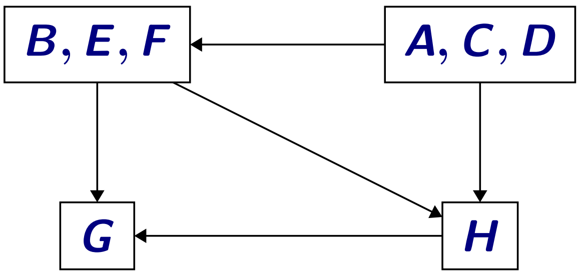
Equivalence classes of C: *strong connected components* of G.
They *partition* the vertices of G.
$\mathrm{SCC}(u)$: strongly connected component containing u.

Structure of a Directed Graph



Graph G



Graph of SCCs G^{SCC}

Reminder

G^{SCC} is created by collapsing every strong connected component to a single vertex.

Proposition

For a directed graph G, its meta-graph G^{SCC} is a DAG.

DAG Properties

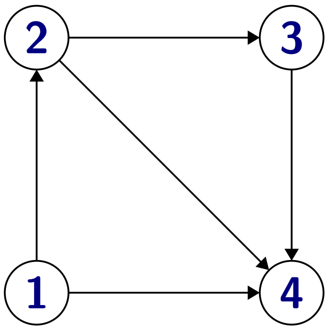
Proposition

Every DAG G *has at least one source and at least one sink.*

Proposition

A directed graph G *can be topologically ordered iff it is a* DAG.

Topological Ordering/Sorting



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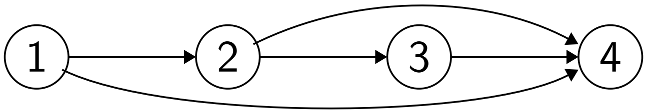

Topological Ordering of G

Definition

A **topological ordering**/**topological sorting** of $G = (V, E)$ is an ordering $\prec$ on V such that if $(u, v) \in E$ then $u \prec v$.

Informal equivalent definition:

One can order the vertices of the graph along a line (say the x-axis) such that all edges are from left to right.

DAGs and Topological Sort

What does it mean?

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Consider a dependency graph.

Topological ordering

Find an order of events in which all dependencies are satisfied.

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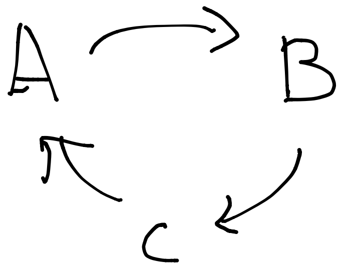
Case 2: Circular dependence.



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Consider a dependency graph.

Topological ordering

Find an order of events in which all dependencies are satisfied.

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Case 2: Circular dependence.

Application: Given pairwise ranking, find an overall ranking that satisfies all pairwise ranking.

Part IV

Graph Search

Basic Search

Given $G = (V, E)$ and vertex $u \in V$. Let $n = |V|$.

```
Explore(G,u):
    array Visited[1..n]
    Initialize:  Set Visited[i] = FALSE for 1 ≤ i ≤ n
    List:  ToExplore, S
    Add u to ToExplore and to S, Visited[u] = TRUE
    while (ToExplore is non-empty) do
        Remove node x from ToExplore
        for each edge (x, y) in Adj(x) do
            if (Visited[y] == FALSE)
                Visited[y] = TRUE
                Add y to ToExplore
                Add y to S
    Output S
```

Running time: O(n+m)

Properties of Basic Search

Proposition

On an undirected graph, **Explore**(G, u) *terminates with* $S = con(u)$.

Proposition

On a directed graph, **Explore**(G, u) *terminates with* $S = rch(u)$.

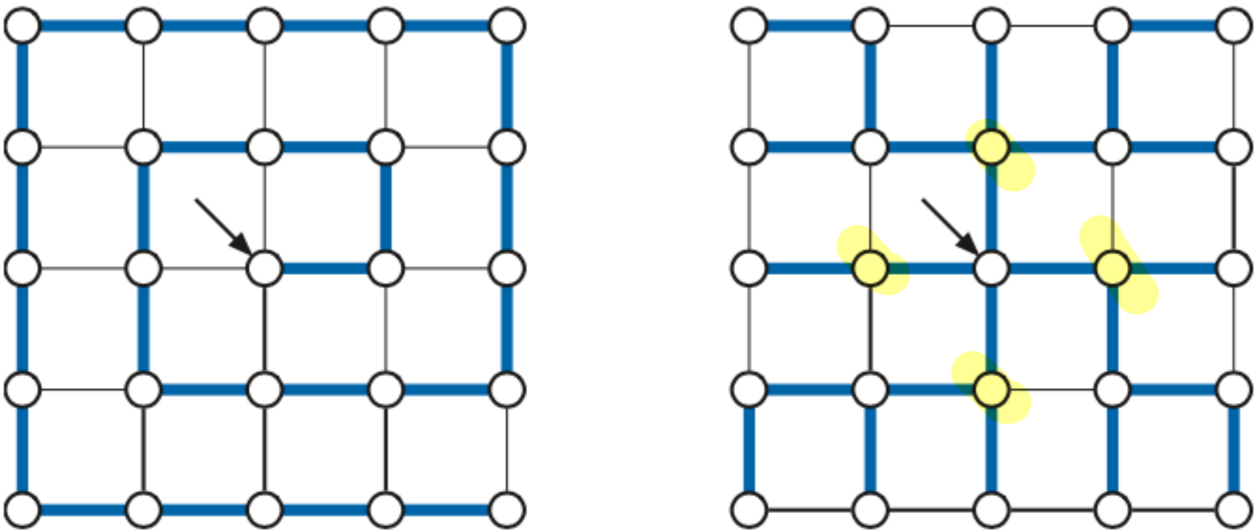
Properties of Basic Search

DFS and **BFS** are special case of BasicSearch.

1. Depth First Search (**DFS**): use stack data structure to implement the list ***ToExplore***
2. Breadth First Search (**BFS**): use queue data structure to implementing the list ***ToExplore***

Spanning tree

A depth-first and breadth-first spanning tree.



Algorithms via Basic Search-II

1. Given G and u, compute all v that can reach u, that is all v such that $u \in \text{rch}(v)$.

Definition (Reverse graph.)

Given $G = (V, E)$, G^{rev} is the graph with edge directions reversed $G^{rev} = (V, E')$ where $E' = \{(y, x) \mid (x, y) \in E\}$

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Given $G = (V, E)$, G^{rev} is the graph with edge directions reversed $G^{rev} = (V, E')$ where $E' = \{(y, x) \mid (x, y) \in E\}$

Compute $\text{rch}(u)$ in G^{rev}!

1. **Running time:** $O(n + m)$ to obtain G^{rev} from G and $O(n + m)$ time to compute $\text{rch}(u)$ via Basic Search.

Algorithms via Basic Search - III

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$\mathrm{SCC}(G, u) = \mathrm{rch}(G, u) \cap \mathrm{rch}(G^{rev}, u)$

Algorithms via Basic Search - III

$\mathrm{SCC}(G, u) = \{v \mid u \text{ is strongly connected to } v\}$

1. Find the strongly connected component containing node u. That is, compute $\mathrm{SCC}(G, u)$.

$\mathrm{SCC}(G, u) = \mathrm{rch}(G, u) \cap \mathrm{rch}(G^{rev}, u)$

Hence, $\mathrm{SCC}(G, u)$ can be computed with $\mathit{Explore}(G, u)$ and $\mathit{Explore}(G^{rev}, u)$. Total $O(n + m)$ time.

Algorithms via Basic Search - IV

1. Is G strongly connected?

Algorithms via Basic Search - IV

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Pick arbitrary vertex u. Check if $\mathrm{SCC}(G, u) = V$.

DFS with Visit Times

Keep track of when nodes are visited.

```
DFS(G)
    for all u ∈ V(G) do
        Mark u as unvisited
    T is set to ∅
    time = 0
    while ∃unvisited u do
        DFS(u)
    Output T
```

```
DFS(u)
    Mark u as visited
    pre(u) = ++time
    for each uv in Out(u) do
        if v is not marked then
            add edge uv to T
            DFS(v)
    post(u) = ++time
```

An Edge in DAG

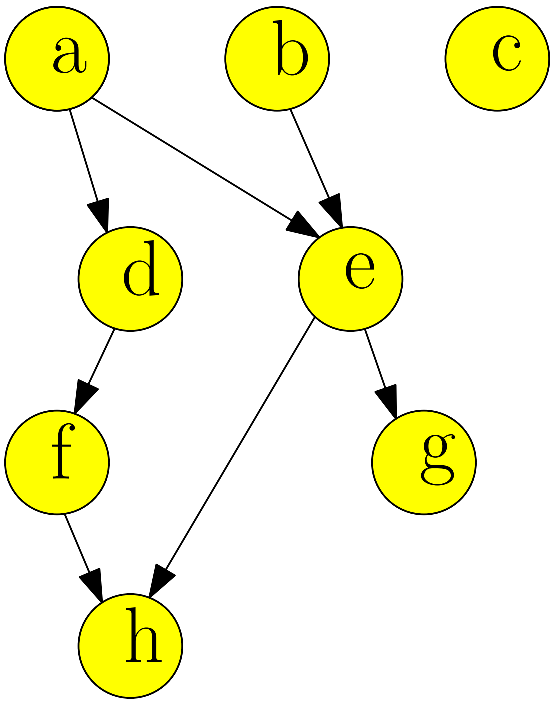
Proposition

If G is a DAG *and* $\mathrm{post}(u) < \mathrm{post}(v)$, *then* (u, v) *is not in* G.
i.e., for all edges (u, v) *in a* DAG, $\mathrm{post}(u) > \mathrm{post}(v)$.

post $>$ $>$ $>$ $>$

u_1 u_2 u_3 $\ldots$ u_n

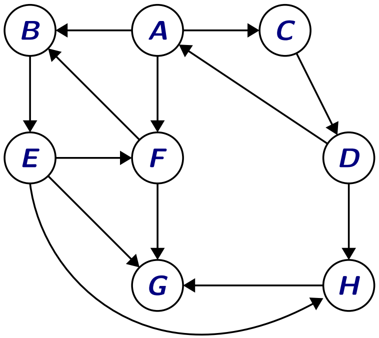
Reverse post-order is topological order



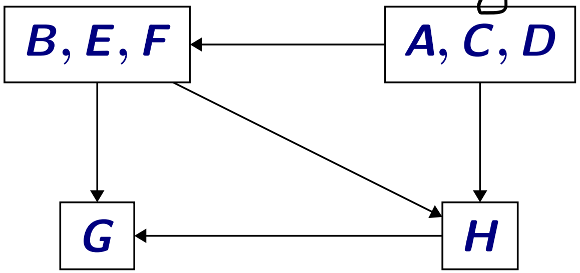
Sort SCCs

The SCCs are topologically sorted by arranging them in decreasing order of their highest post number.

highest post



Graph G



Graph of SCCs G^{SCC}

DFS post.

Linear Time Algorithm

...for computing the strong connected components in **G**

```
do DFS(G^rev) and output vertices in decreasing post order.
Mark all nodes as unvisited
for each u in the computed order do
    if u is not visited then
        DFS(u)
        Let S_u be the nodes reached by u
        Output S_u as a strong connected component
        Remove S_u from G
```

Theorem

Algorithm runs in time $O(m + n)$ and correctly outputs all the SCCs of G.

Using DAG and SCC

A node u is good if it can reach every node in V. Describe a linear-time algorithm to find if there is a good node in G.

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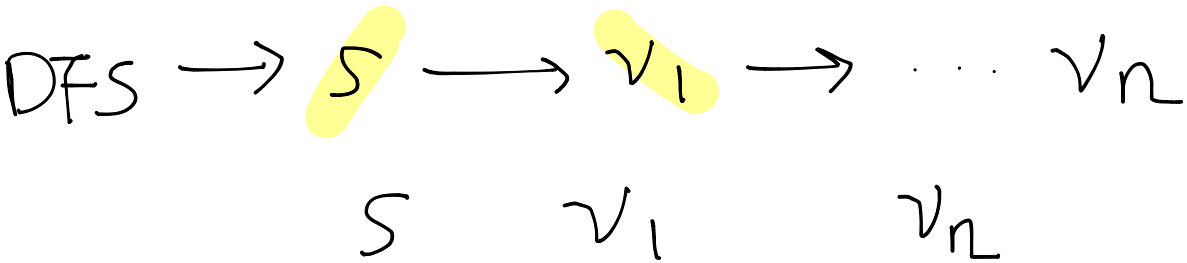

Using DAG and SCC

A node u is good if it can reach every node in V. Describe a linear-time algorithm to find if there is a good node in G.

1. First consider a DAG.
2. For any directed graph, construct the meta-graph G^{SCC}, which is a DAG.

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1. First consider a DAG.
2. For any directed graph, construct the meta-graph G^{SCC}, which is a DAG.
3. The good node, if exists, has to be in the source SCC.

u

Part V

Shortest Path in Graphs

Breadth First Search (BFS)

Overview

(A) **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a **queue**.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex s (the start vertex).

BFS finds *shortest distance* starting from s on unweighted graphs.

Non-negative edge length: Dijkstra

```
Initialize for each node v, dist(s, v) = ∞
Initialize X = {s},
for i = 2 to |V| do
    (* Invariant:  X contains the i − 1 closest nodes to s *)
    Among nodes in V − X, find the node v that is the
            i'th closest to s
    Update dist(s, v)
    X = X ∪ {v}
```

Dijkstra's Algorithm using Priority Queues

```
Q ← makePQ()
insert(Q, (s, 0))
for each node u ≠ s do
    insert(Q, (u, ∞))
    (* Invariant: X contains the i − 1 closest nodes to s *)
    (* Invariant: d′(s, u) is shortest path distance from s to u
     using only X as intermediate nodes*)
X ← ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    X = X ∪ {v}
    for each u in Adj(v) do
        decreaseKey(Q, (u, min(dist(s, u), dist(s, v) + ℓ(v, u)))).
```

Running time: $O((m+n)\log n)$ with heaps and $O(m + n\log n)$ with advanced priority queues.

One negative edge: Use Dijkstra

Compute the shortest path from s to t on a graph with exactly one negative edge $x \rightarrow y$.

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    1. Remove the negative edge: G'.

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1. Detect if there is a negative length cycle.
    1. Remove the negative edge: G'.
    2. Compute the shortest distance $y \to x$ on G'.

One negative edge: Use Dijkstra

Compute the shortest path from s to t on a graph with exactly one negative edge $x \to y$.

1. Detect if there is a negative length cycle.
   1. Remove the negative edge: G'.
   2. Compute the shortest distance $y \to x$ on G'.
2. Suppose no negative length cycle, find shortest distance by

$$dist(s,t) = \min \left\{ \begin{array}{c} dist'(s,t) \\ dist'(s,u) + w(u \to v) + dist'(v,t) \end{array} \right\}$$

Negative-length edges: Bellman-Ford Algorithm

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
    for each v ∈ V do
        for each edge (u, v) ∈ In(v) do
            d(v) = min{d(v), d(u) + ℓ(u, v)}

for each v ∈ V do
    dist(s, v) ← d(v)
```

Running time: $O(mn)$

Bellman-Ford: Negative Cycle Detection

Check if distances change in iteration n.

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
        for each v ∈ V do
            for each edge (u, v) ∈ In(v) do
                d(v) = min{d(v), d(u) + ℓ(u, v)}
(* One more iteration to check if distances change *)
for each v ∈ V do
    for each edge (u, v) ∈ In(v) do
        if (d(v) > d(u) + ℓ(u, v))
            Output ''Negative Cycle''

for each v ∈ V do
        dist(s, v) ← d(v)
```

Algorithm for DAGs

Observation:

1. shortest path from s to v_i cannot use any node from $v_{i+1}, \ldots, v_n$
2. can find shortest paths in topological sort order.

Algorithm for DAGs

Let $s = v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of G

```
for i = 1 to n do
        d(s, v_i) = ∞
d(s, s) = 0

for i = 1 to n − 1 do
        for each edge (v_i, v_j) in Out(v_i) do
                d(s, v_j) = min{d(s, v_j), d(s, v_i) + ℓ(v_i, v_j)}

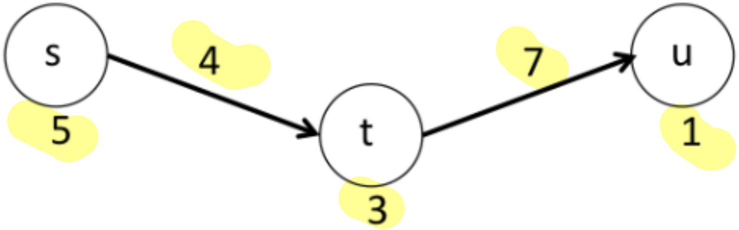
return d(s, ·) values computed
```

Running time: $O(m + n)$ time algorithm! Works for negative edge lengths and hence can find *longest* paths in a DAG.

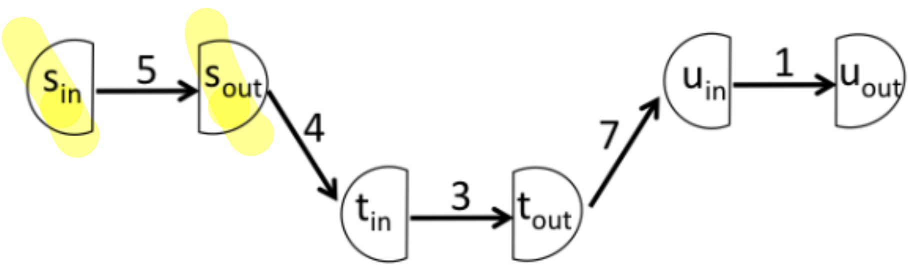
Part VI

Graph reduction and tricks

Split nodes



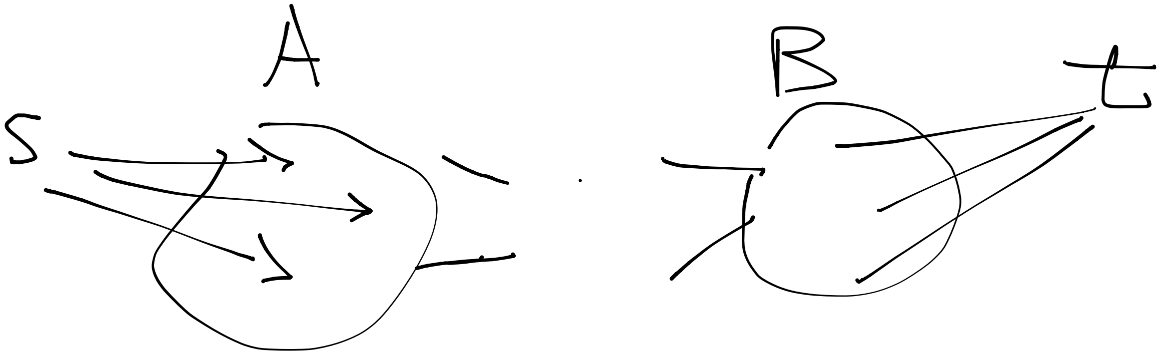
original graph
with vertex weights



new graph
with only edge weights

Add nodes

Given a graph $G = (V, E)$ and two disjoint sets of nodes $A, B \subset V$, is there a path from some node in A to some node in B?



Add nodes

Given a graph $G = (V, E)$ and two disjoint sets of nodes $A, B \subset V$, is there a path from some node in A to some node in B?

Connect s to each node in A, and t to each node in B. This becomes the basic $s - t$ reachability problem.

DP on graphs

Q: How to compute the shortest distance between s and t with at most k hops?

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Ans: We arrived at Bellman-Ford by considering the shortest distance with at most k hops.

$d(u, k)$

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Ans: We arrived at Bellman-Ford by considering the shortest distance with at most k hops.

Q: A subset of risky ~~nodes~~ edges $E' \subset E$. Find shortest path from s with at most h risky edges.

Ans: Use Bellman-Ford style DP. Consider which $u \to v$ edge to include for each v.

$d(s, u_1) \to v \quad + w(u_1, v)$

$d(s, u_2)$

$d(s, u_3)$

$\vdots$

DP on graphs

Q: A subset of risky nodes $E' \subset E$. Find shortest path from s with at most h risky edges.

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$$d(v,i,j) = \min \begin{cases} d(v,i-1,j) \\ d(v,i,j-1) \\ \min_{(u,v)\in E'} \; d(u,i-1,j-1) + \ell(u,v) \\ \min_{(u,v)\in E-E'} \; d(u,i-1,j) + \ell(u,v) \end{cases}$$

risky

not risky

$u \to v$

DP on graphs

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Base case: Use Bellman-Ford to compute $d(v,i,0)$, shortest distance on G' with no risky edge.
Running time: $O(mnk)$.

Layering

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→ remove risky

1. Create $h + 1$ copies of G': $G_0, G_1, \ldots, G_h$

Layering

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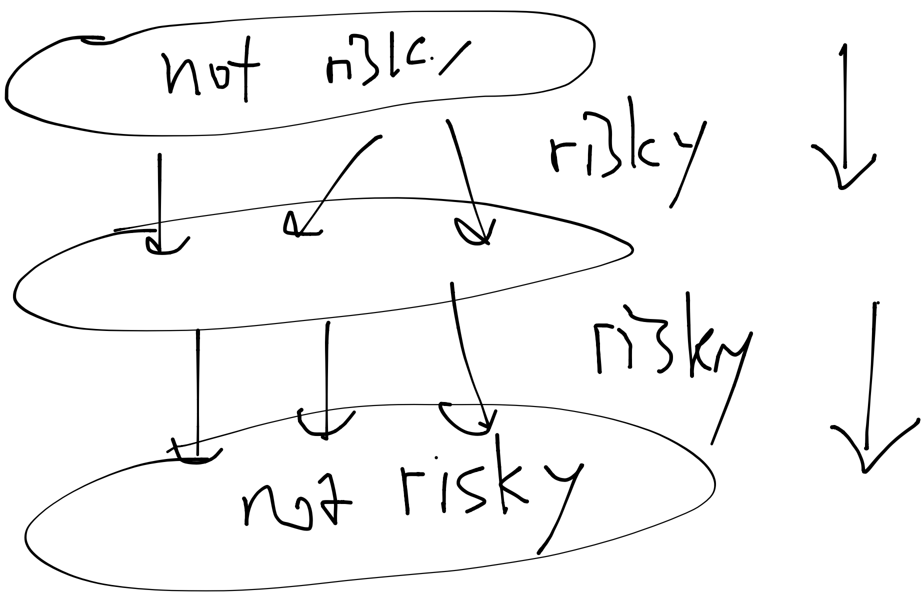
1. Create $h+1$ copies of G': $G_0, G_1, \dots, G_h$
2. Include a directed edge from vertex u in G_i to vertex v in G_{i+1} if (u, v) is a risky edge in G.



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Running time: $O(mk + nk \log(nk))$