# Midterm 2 review

Lecture 22

# Part I

# Recursion: Divide and Conquer

# Recursion types

1. **Divide and Conquer**: Problem reduced to multiple **independent** sub-problems.

   Examples: Binary search, Merge sort, quick sort, multiplication, median selection.

   Each sub-problem is a fraction smaller.

# Binary Search

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2. Recurrence tree

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2. Recurrence tree
3. Which condition to check?

# Binary Search

Suppose you are given two sorted arrays $A[1..n]$ and $B[1..n]$ containing distinct integers. Describe a fast algorithm to find the median (meaning the $n$th smallest element) of the union $A \cup B$. For example, given the input

$$A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20]$$

$$B[1..8] = [2, 4, 5, 8, 17, 19, 21, 23]$$

your algorithm should return the integer $9$.

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$$B[1 \,..\, 8] = [2, 4, 5, 8, 17, 19, 21, 23]$$

your algorithm should return the integer $9$.

Compare the two medians.

# Binary Search

```
MEDIAN(A[1..n], B[1..n]):
  if n < 10^100
      use brute force
  else if A[n/2] > B[n/2]
      return MEDIAN(A[1..n/2], B[n/2+1..n])
  else
      return MEDIAN(A[n/2+1..n], B[1..n/2])
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Because we discard the same number of elements from each array, the median of the remaining subarrays is the median of the original $A \cup B$.

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$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

# Karatsuba's Algorithm

$$\begin{aligned} xy &= (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \\ &= 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R \end{aligned}$$

Gauss trick: $x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$

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Recursively compute only $x_L y_L, x_R y_R, (x_L + x_R)(y_L + y_R)$.

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## Time Analysis

Running time is given by

$$T(n) = 3T(n/2) + O(n) \qquad T(1) = O(1)$$

which means

# Karatsuba's Algorithm

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## Time Analysis

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which means $T(n) = O(n^{\log_2 3}) = O(n^{1.585})$

# Recursion tree analysis

# Selecting in Unsorted Lists

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1. One-armed Quick-sort
2. With a good pivot (median of the medians)

$$T(n) \leq T(\lceil n/5 \rceil) + T(\lceil 7n/10 \rceil) + O(n)$$

and

$$T(n) = O(1) \qquad n < 10$$

# Recursion tree analysis

# Part II

# Dynamic programming

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3. **Dynamic Programming**: Smart recursion with memoization

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**Max-Weight Independent Set in Trees:** Two possibilities: Include the root or not

# How to design DP algorithms

1. Find a “smart” recursion (The hard part)
    1. Formulate the sub-problem
    2. so that the number of distinct subproblems is small; polynomial in the original problem size.

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1. Find a “smart” recursion (The hard part)
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2. Memoization
   1. Identify distinct subproblems
   2. Choose a memoization data structure
   3. Identify dependencies and find a good evaluation order
   4. An iterative algorithm replacing recursive calls with array lookups
   5. Further optimize space

# Which data structure?

- Text segmentation, suffix, 1-D array
- Longest increasing subsequence, suffix+index, 2-D array
- Edit distance, two prefixes, 2-D array
- Max-Weight Independent Set in Trees, tree

# Part III

# Graphs

# Path and cycle

A path is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k-1$. The length of the path is $k-1$ (the number of edges in the path) and the path is from $v_1$ to $v_k$.
Note: a single vertex $u$ is a path of length $0$.

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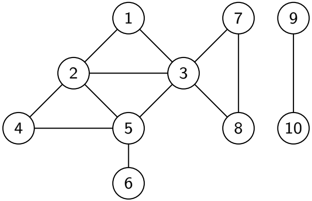
Note: a single vertex $u$ is a path of length $0$.

A cycle is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \le i \le k-1$ and $\{v_1, v_k\} \in E$. Single vertex not a cycle according to this definition.

# Connectivity on Undirected Graphs

Given a graph $G = (V, E)$:

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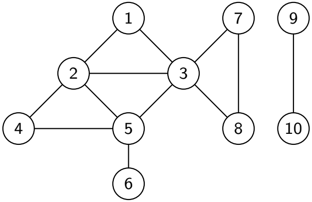


A vertex $u$ is connected to $v$ if there is a path from $u$ to $v$.

The connected component of $u$, $\text{con}(u)$, is the set of all vertices connected to $u$.

# Directed Connectivity

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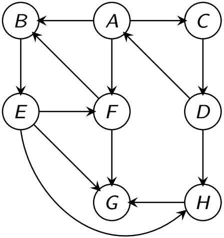


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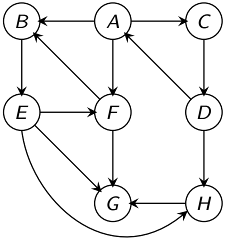


A vertex $u$ can reach $v$ if there is a path from $u$ to $v$.

Let $\mathbf{rch}(u)$ be the set of all vertices reachable from $u$.

Asymmetricity: $D$ can reach $B$ but $B$ cannot reach $D$

# Connectivity and Strong Connected Components

## Definition

Given a directed graph $G$, $u$ is strongly connected to $v$ if $u$ can reach $v$ *and* $v$ can reach $u$. In other words $v \in \mathrm{rch}(u)$ and $u \in \mathrm{rch}(v)$.

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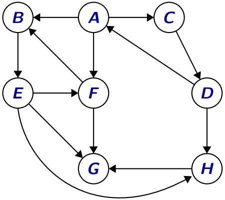
## Proposition

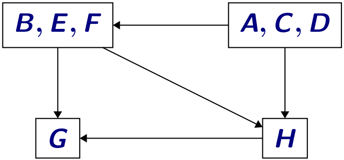
*$C$ is an equivalence relation, that is reflexive, symmetric and transitive.*

Equivalence classes of $C$: *strong connected components* of $G$.
They *partition* the vertices of $G$.
$\text{SCC}(u)$: strongly connected component containing $u$.

# Structure of a Directed Graph




Graph G




Graph of SCCs $G^{\mathrm{SCC}}$

## Reminder

$G^{\mathrm{SCC}}$ is created by collapsing every strong connected component to a single vertex.

## Proposition

*For a directed graph G, its meta-graph $G^{\mathrm{SCC}}$ is a DAG.*

# DAG Properties

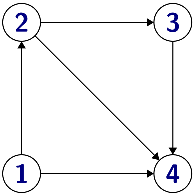
**Proposition**

*Every DAG $G$ has at least one source and at least one sink.*

**Proposition**

*A directed graph $G$ can be topologically ordered iff it is a DAG.*

# Topological Ordering/Sorting

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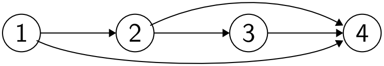


Topological Ordering of G

## Definition

A **topological ordering**/**topological sorting** of $G = (V, E)$ is an ordering $\prec$ on $V$ such that if $(u, v) \in E$ then $u \prec v$.

## Informal equivalent definition:

One can order the vertices of the graph along a line (say the $x$-axis) such that all edges are from left to right.

# DAGs and Topological Sort

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Consider a dependency graph.

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Find an order of events in which all dependencies are satisfied.

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Consider a dependency graph.

**Topological ordering**

Find an order of events in which all dependencies are satisfied.

Case 1: DAG. Heat a pizza $\longrightarrow$ eat the pizza, have a Coke.
Case 2: Circular dependence.

Application: Given pairwise ranking, find an overall ranking that satisfies all pairwise ranking.

# Part IV

# Graph Search

# Basic Search

Given $G = (V, E)$ and vertex $u \in V$. Let $n = |V|$.

```
Explore(G, u):
    array Visited[1..n]
    Initialize:  Set Visited[i] = FALSE for 1 ≤ i ≤ n
    List:  ToExplore, S
    Add u to ToExplore and to S, Visited[u] = TRUE
    while (ToExplore is non-empty) do
        Remove node x from ToExplore
        for each edge (x, y) in Adj(x) do
            if (Visited[y] == FALSE)
                Visited[y] = TRUE
                Add y to ToExplore
                Add y to S
    Output S
```

Running time: O(n+m)

# Properties of Basic Search

**Proposition**

*On an undirected graph,* **Explore**$(G, u)$ *terminates with* $S = con(u)$.

**Proposition**

*On a directed graph,* **Explore**$(G, u)$ *terminates with* $S = rch(u)$.

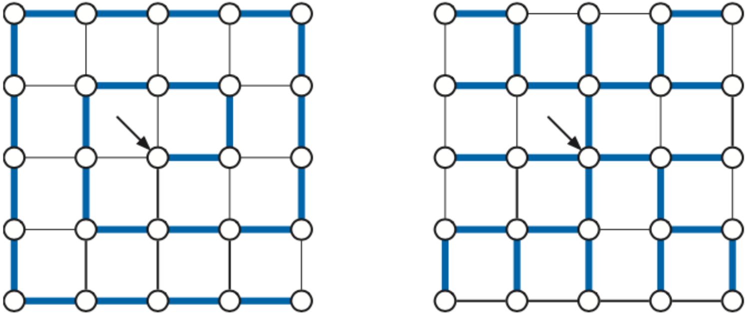
# Properties of Basic Search

**DFS** and **BFS** are special case of BasicSearch.

1. Depth First Search (**DFS**): use stack data structure to implement the list ***ToExplore***
2. Breadth First Search (**BFS**): use queue data structure to implementing the list ***ToExplore***

# Spanning tree

A depth-first and breadth-first spanning tree.



# Algorithms via Basic Search-II

1. Given $G$ and $u$, compute all $v$ that can reach $u$, that is all $v$ such that $u \in \text{rch}(v)$.

**Definition (Reverse graph.)**

Given $G = (V, E)$, $G^{rev}$ is the graph with edge directions reversed $G^{rev} = (V, E')$ where $E' = \{(y, x) \mid (x, y) \in E\}$

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Compute $\text{rch}(u)$ in $G^{rev}$!

1. **Running time:** $O(n + m)$ to obtain $G^{rev}$ from $G$ and $O(n + m)$ time to compute $\text{rch}(u)$ via Basic Search.

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# Algorithms via Basic Search - III

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1. Find the strongly connected component containing node $u$. That is, compute $\text{SCC}(G, u)$.

$\text{SCC}(G, u) = \text{rch}(G, u) \cap \text{rch}(G^{rev}, u)$

Hence, $\text{SCC}(G, u)$ can be computed with $\textit{Explore}(G, u)$ and $\textit{Explore}(G^{rev}, u)$. Total $O(n + m)$ time.

# Algorithms via Basic Search - IV

1. Is $G$ strongly connected?

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1. Is $G$ strongly connected?

Pick arbitrary vertex $u$. Check if $\mathrm{SCC}(G, u) = V$.

# DFS with Visit Times

Keep track of when nodes are visited.

```
DFS(G)
    for all u ∈ V(G) do
        Mark u as unvisited
    T is set to ∅
    time = 0
    while ∃unvisited u do
        DFS(u)
    Output T
```

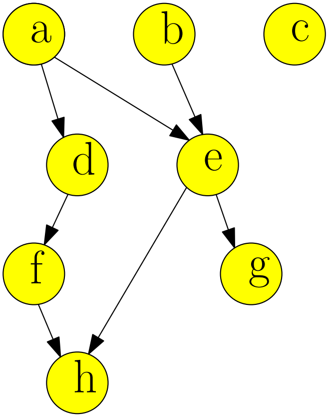
```
DFS(u)
    Mark u as visited
    pre(u) = ++time
    for each uv in Out(u) do
        if v is not marked then
            add edge uv to T
            DFS(v)
    post(u) = ++time
```

# An Edge in DAG

## Proposition

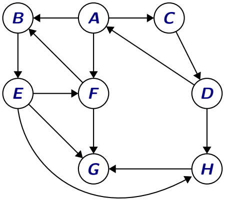
*If* $G$ *is a* $\mathrm{DAG}$ *and* $\mathrm{post}(u) < \mathrm{post}(v)$*, then* $(u, v)$ *is not in* $G$.
*i.e., for all edges* $(u, v)$ *in a* $\mathrm{DAG}$, $\mathrm{post}(u) > \mathrm{post}(v)$.

# Reverse post-order is topological order

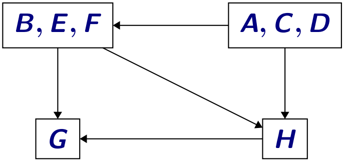


# Sort SCCs

The SCCs are topologically sorted by arranging them in decreasing order of their highest post number.




Graph G




Graph of SCCs $G^{SCC}$

# Linear Time Algorithm

...for computing the strong connected components in **G**

```
do DFS(G^rev) and output vertices in decreasing post order.
Mark all nodes as unvisited
for each u in the computed order do
    if u is not visited then
        DFS(u)
        Let S_u be the nodes reached by u
        Output S_u as a strong connected component
        Remove S_u from G
```

## Theorem

*Algorithm runs in time $O(m + n)$ and correctly outputs all the SCCs of $G$.*

# Using DAG and SCC

A node $u$ is good if it can reach every node in $V$. Describe a linear-time algorithm to find if there is a good node in $G$.

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1. First consider a DAG.
2. For any directed graph, construct the meta-graph $G^{SCC}$, which is a DAG.
3. The good node, if exists, has to be in the source SCC.

# Part V

# Shortest Path in Graphs

# Breadth First Search (BFS)

## Overview

1. **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a **queue**.
2. It processes the vertices in the graph in the order of their shortest distance from the vertex $s$ (the start vertex).

**BFS** finds *shortest distance* starting from $s$ on unweighted graphs.

# Non-negative edge length: Dijkstra

```
Initialize for each node v, dist(s, v) = ∞
Initialize X = {s},
for i = 2 to |V| do
    (* Invariant:  X contains the i − 1 closest nodes to s *)
    Among nodes in V − X, find the node v that is the
            i'th closest to s
    Update dist(s, v)
    X = X ∪ {v}
```

# Dijkstra's Algorithm using Priority Queues

```
Q ← makePQ()
insert(Q, (s, 0))
for each node u ≠ s do
    insert(Q, (u,∞))
    (* Invariant:  X contains the i − 1 closest nodes to s *)
    (* Invariant:  d′(s, u) is shortest path distance from s to u
     using only X as intermediate nodes*)
X ← ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    X = X ∪ {v}
    for each u in Adj(v) do
        decreaseKey(Q, (u, min(dist(s, u), dist(s, v) + ℓ(v, u)))).
```

Running time: $O((m+n)\log n)$ with heaps and $O(m+n\log n)$ with advanced priority queues.

# One negative edge: Use Dijkstra

Compute the shortest path from $s$ to $t$ on a graph with exactly one negative edge $x \rightarrow y$.

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Compute the shortest path from $s$ to $t$ on a graph with exactly one negative edge $x \to y$.

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    1. Remove the negative edge: $G'$.

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1. Detect if there is a negative length cycle.
   1. Remove the negative edge: $G'$.
   2. Compute the shortest distance $y \to x$ on $G'$.

# One negative edge: Use Dijkstra

Compute the shortest path from $s$ to $t$ on a graph with exactly one negative edge $x \to y$.

1. Detect if there is a negative length cycle.
   1. Remove the negative edge: $G'$.
   2. Compute the shortest distance $y \to x$ on $G'$.
2. Suppose no negative length cycle, find shortest distance by

$$dist(s,t) = \min \left\{ \begin{array}{c} dist'(s,t) \\ dist'(s,u) + w(u \to v) + dist'(v,t) \end{array} \right\}$$

# Negative-length edges: Bellman-Ford Algorithm

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
        for each v ∈ V do
            for each edge (u, v) ∈ In(v) do
                d(v) = min{d(v), d(u) + ℓ(u, v)}

for each v ∈ V do
        dist(s, v) ← d(v)
```

Running time: $O(mn)$

# Bellman-Ford: Negative Cycle Detection

Check if distances change in iteration $n$.

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
        for each v ∈ V do
            for each edge (u, v) ∈ In(v) do
                d(v) = min{d(v), d(u) + ℓ(u, v)}
(* One more iteration to check if distances change *)
for each v ∈ V do
    for each edge (u, v) ∈ In(v) do
        if (d(v) > d(u) + ℓ(u, v))
            Output ``Negative Cycle''

for each v ∈ V do
        dist(s, v) ← d(v)
```

# Algorithm for DAGs

**Observation:**

1. shortest path from $s$ to $v_i$ cannot use any node from $v_{i+1}, \ldots, v_n$
2. can find shortest paths in topological sort order.

# Algorithm for DAGs

Let $s = v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of $G$

```
for i = 1 to n do
        d(s, v_i) = ∞
d(s, s) = 0

for i = 1 to n − 1 do
        for each edge (v_i, v_j) in Out(v_i) do
              d(s, v_j) = min{d(s, v_j), d(s, v_i) + ℓ(v_i, v_j)}

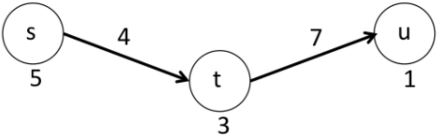
return d(s, ·) values computed
```

Running time: $O(m + n)$ time algorithm! Works for negative edge lengths and hence can find *longest* paths in a DAG.

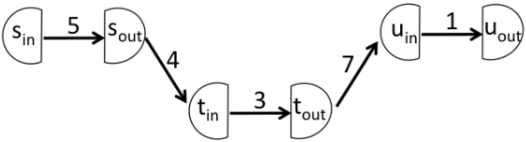
# Part VI

## Graph reduction and tricks

# Split nodes




original graph
with vertex weights




new graph
with only edge weights

# Add nodes

Given a graph $G = (V, E)$ and two disjoint sets of nodes $A, B \subset V$, is there a path from some node in $A$ to some node in $B$?

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Given a graph $G = (V, E)$ and two disjoint sets of nodes $A, B \subset V$, is there a path from some node in $A$ to some node in $B$?

Connect $s$ to each node in $A$, and $t$ to each node in $B$. This becomes the basic $s - t$ reachability problem.

# DP on graphs

Q: How to compute the shortest distance between $s$ and $t$ with at most $k$ hops?

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Q: A subset of risky nodes $E' \subset E$. Find shortest path from $s$ with at most $h$ risky edges.

Ans: Use Bellman-Ford style DP. Consider which $u \rightarrow v$ edge to include for each $v$.

# DP on graphs

Q: A subset of risky nodes $E' \subset E$. Find shortest path from $s$ with at most $h$ risky edges.

Ans: Use Bellman-Ford style DP. Consider which $u \to v$ edge to include for each $v$. Remove the risky nodes to form $G'$.

# DP on graphs

Q: A subset of risky nodes $E' \subset E$. Find shortest path from $s$ with at most $h$ risky edges.

Ans: Use Bellman-Ford style DP. Consider which $u \to v$ edge to include for each $v$. Remove the risky nodes to form $G'$.

$$d(v,i,j) = \min \begin{cases} d(v,i-1,j) \\ d(v,i,j-1) \\ \min_{(u,v)\in E'} \; d(u,i-1,j-1) + \ell(u,v) \\ \min_{(u,v)\in E-E'} \; d(u,i-1,j) + \ell(u,v) \end{cases}$$

# DP on graphs

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Base case: Use Bellman-Ford to compute $\mathbf{d(v,i,0)}$, shortest distance on $G'$ with no risky edge.
Running time: $\mathbf{O(mnk)}$.

# Layering

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4. Run Dijkstra's algorithm on this new graph, from vertex $s_0$, the copy of $s$ in $G_0$, to $v_0, \ldots, v_h$ be the corresponding vertices in copies $G_0, \ldots, G_h$.

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6. the distance from $s$ to $v$ in the original graph that uses at most $h$ risky edges is just $\min_{0 \leq i \leq h} d(s_0, v_i)$.

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Q: A subset of risky nodes $E' \subset E$. Find shortest path from $s$ with at most $h$ risky edges.

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Running time: $O(mk + nk \log(nk))$