Pre-lecture brain teaser

Is the following language regular? Either way, prove it.

$L = \{\text{strings of properly matched open and closing parentheses}\}$

CS/ECE-374: Lecture 8 - Context-Free languages and Turing Machines

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Chat moderator: Samir Khan

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University of Illinois at Urbana-Champaign

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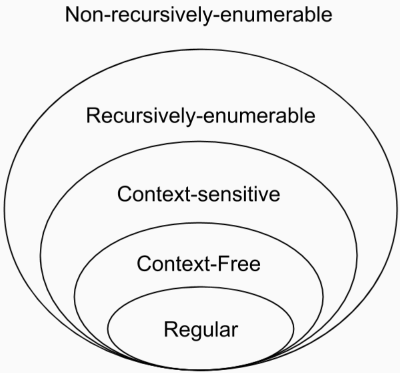
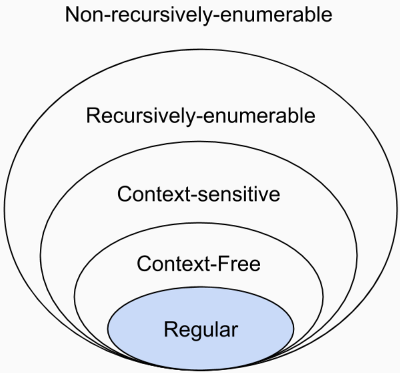
Larger world of languages!

Chomsky Hierarchy



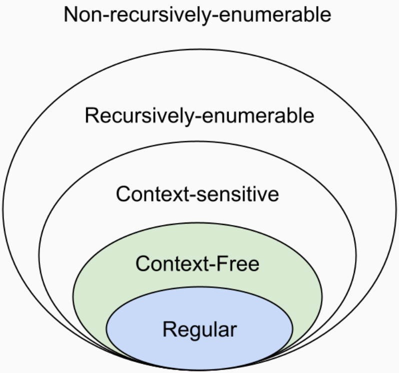
Remember our hierarchy of languages

Chomsky Hierarchy



You've mastered regular expressions.

Chomsky Hierarchy



Now what about the next level up?

Context-Free Languages

Example

- $V = \{S\}$
- $T = \{a, b\}$
- $P = \{S \to \epsilon \mid a \mid b \mid aSa \mid bSb\}$
  (abbrev. for $S \to \epsilon, S \to a, S \to b, S \to aSa, S \to bSb$)

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What strings can S generate like this?

Context Free Grammar (CFG) Definition

Definition
A CFG is a quadruple $G = (V, T, P, S)$

- V is a finite set of non-terminal symbols

$$G = \Big(\quad \text{Variables}, \quad \text{Terminals}, \quad \text{Productions}, \quad \text{Start var} \quad \Big)$$

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  where $A \in V$ and α is a string in $(V \cup T)^*$.
  Formally, $P \subset V \times (V \cup T)^*$.

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Example formally...

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Examples

$L = \{0^n 1^n \mid n \geq 0\}$

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The language generated by CFG $G = (V, T, P, S)$ is denoted by $L(G)$ where $L(G) = \{w \in T^* \mid S \rightsquigarrow^* w\}$.

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Definition
A language L is context free (CFL) if it is generated by a context free grammar. That is, there is a CFG G such that $L = L(G)$.

Example

$L = \{0^n 1^n \mid n \geq 0\}$

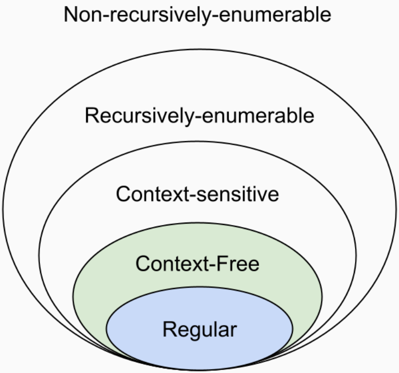
$S \to \epsilon \mid 0S1$

$L = \{0^n 1^m \mid m > n\}$

$L = \left\{ w \in \{(,)\}^* \,\middle|\, w \text{ is properly nested string of parenthesis} \right\}.$

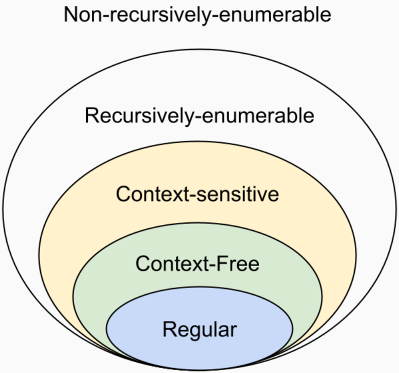
Context-Sensitive Langauges

Chomsky Hierarchy



Now that we ~~mastered~~ acknowledged Context-Free Languages.....

Chomsky Hierarchy



On to the next one.....

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$$\begin{aligned} S &\rightsquigarrow aAbc \rightsquigarrow abAc \rightsquigarrow abBbcc \rightsquigarrow aBbbcc \rightsquigarrow aaAbbcc \rightsquigarrow aabAbcc \\ &\rightsquigarrow aabbAcc \rightsquigarrow aabbBbccc \rightsquigarrow aabBbbccc \rightsquigarrow aaBbbbccc \\ &\rightsquigarrow aaabbbccc \end{aligned}$$

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Turing Machines

"Most General" computer?

- DFAs are simple model of computation.
- Accept only the regular languages.
- Is there a kind of computer that can accept any language, or compute any function?
- Recall counting argument. Set of all languages: $\{L \mid L \subseteq \{0,1\}^*\}$ is ~~countably infinite~~ / uncountably infinite

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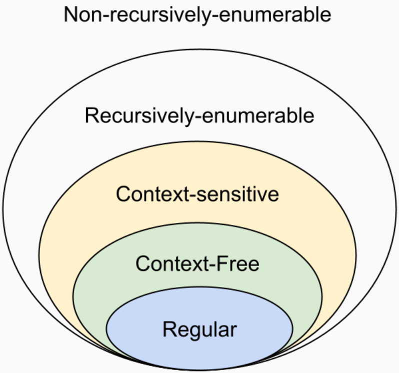
- Set of all programs: $\{P \mid P$ is a finite length computer program$\}$: is countably infinite / ~~uncountably infinite.~~
- **Conclusion:** There are languages for which there are no programs.

Chomsky Hierarchy



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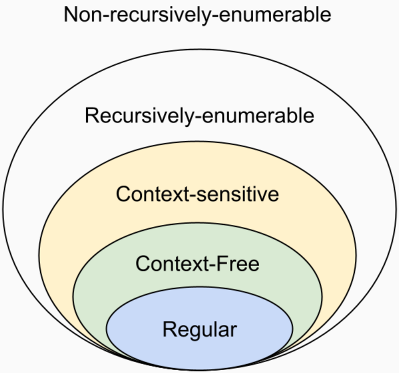
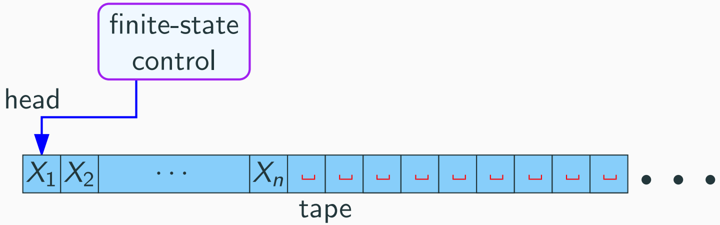


Onto our final class of languages - recursively enumerable (aka Turing-recognizable) languages.

What is a Turing machine

Turing machine



- Input written on (infinite) one sided tape.
- Special blank characters.
- Finite state control (similar to DFA).
- Ever step: Read character under head, write character out, move the head right or left (or stay).

High level goals

- Church-Turing thesis: TMs are the most general computing devices. So far no counter example.
- Every TM can be represented as a string.
- Existence of Universal Turing Machine which is the model/inspiration for stored program computing. UTM can simulate any TM
- Implications for what can be computed and what cannot be computed

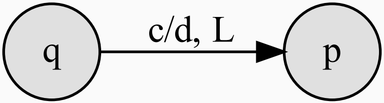
Turing machine: Formal definition

A *Turing machine* is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}})$

- Q: finite set of states.
- Σ: finite input alphabet.
- Γ: finite tape alphabet.
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}, \mathsf{S}\}$: Transition function.
- $q_0 \in Q$ is the initial state.
- $q_{\mathrm{acc}} \in Q$ is the *accepting*/*final* state.
- $q_{\mathrm{rej}} \in Q$ is the *rejecting* state.
- $\sqcup$ or : Special blank symbol on the tape.

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}, \mathsf{S}\}$$

As such, the transition



$$\delta(q, c) = (p, d, \mathsf{L})$$

- q: current state.
- c: character under tape head.
- p: new state.
- d: character to write under tape head
- L: Move tape head left.

Turing machine: Transition function

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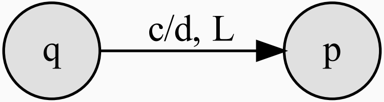

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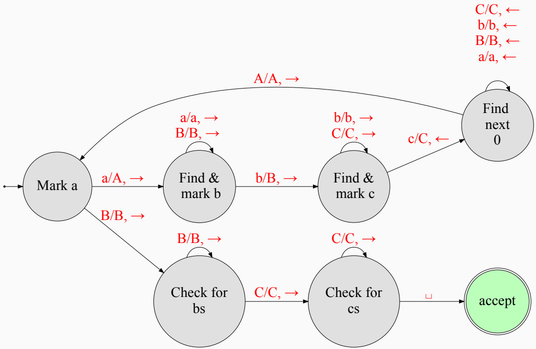
Missing transitions lead to hell state. “Blue screen of death.” “Machine crashes.”

Some examples of Turing machines

Example: Turing machine for $a^n b^n c^n$



Can view this Turing machine in action on `turingmachine.io`!

Languages defined by a Turing machine

Recursive vs. Recursively Enumerable

- *Recursively enumerable* (aka *RE*) languages

$$L = \{L(M) \mid M \text{ some Turing machine}\}.$$

- *Recursive* / *decidable* languages

$$L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}.$$

Recursive vs. Recursively Enumerable

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- Fundamental questions:
    - What languages are RE?
    - Which are recursive?
    - What is the difference?
    - What makes a language decidable?